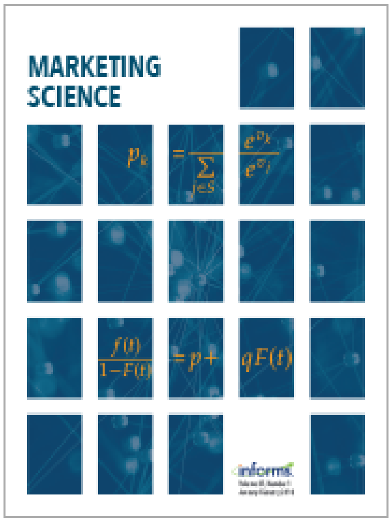

Marketing Science

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To cite this article:
Zijun (June) Shi, Kaifu Zhang, Kannan Srinivasan (2019) Freemium as an Optimal Strategy for Market Dominant Firms. Marketing Science 38(1):150-169. https://doi.org/10.1287/mksc.2018.1109

informs
http://pubsonline.informs.org/journal/mksc/

MARKETING SCIENCE
Vol. 38, No. 1, January–February 2019, pp. 150–169
ISSN 0732-2399 (print), ISSN 1526-548X (online)

Freemium as an Optimal Strategy for Market Dominant Firms

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Received: February 21, 2018
Revised: April 19, 2018
Accepted: April 24, 2018
Published Online in Articles in Advance: February 5, 2019

https://doi.org/10.1287/mksc.2018.1109

Abstract. Despite its immense popularity, the freemium business model remains a complex strategy to master and often a topic of heated debate. Adopting a generalized version of the screening framework, we ask when and why a firm should endogenously offer a zero price on its low-end product when users' product usages generate network externalities on each other. In the standard screening framework without network effects, freemium never emerges as optimal, and the firm always chooses the efficient price point for its low-end product. We show that even with network effects, freemium is typically not optimal. When network effects are identical across products ("symmetric"), the firm has greater incentive to expand its network size and may find it profitable to sell to the low-end customers. However, this does not lead to freemium as an equilibrium strategy. Instead, the firm should offer a low-end product to attract customers, while keeping its price positive. Freemium can only emerge if the high- and low-end products provide different levels of ("asymmetric") marginal network effects. In other words, the firm would set a zero price for its low-end product only if the high-end product provided larger utility gain from an expansion of the firm's user base. In contrast to conventional beliefs, a firm pursuing the freemium strategy might increase the baseline quality on its low-end product above the "efficient" level, which seemingly reduces differentiation.

History: Yuxin Chen served as the senior editor and Debu Purohit served as associate editor for this article.
Supplemental Material: The online appendices are available at https://doi.org/10.1287/mksc.2018.1109.

Keywords: analytic models • game theory • pricing • network externality • product quality

1. Introduction

Over the past years, "freemium" has attracted considerable attention from both practitioners and academics. Many believe that freemium underlies the meteoric rise of companies like Skype and Dropbox, and a horde of startups have jumped on the bandwagon and adopted freemium as their business model of choice. However, successful implementation of the freemium strategy remains challenging. A *Wall Street Journal* report titled "When Freemium Fails" interviewed frustrated entrepreneurs who considered "move (ing) away from freemium" as "the best business decision. . .[they] ever made" (Needleman and Loten 2012). An investment manager at First Round Capital summarized the entrepreneurs' frustration as "too many freemium models have too much free and not enough mium." From online gaming to music streaming, leading companies offer widely divergent opinions on whether and when a free option should be offered at all.[1]

Not only is freemium controversial among practitioners, but it also represents a curious case in the eyes of a theoretician. On the surface, freemium resembles a classic case of product-line screening, wherein a firm offers a menu of products at different prices to segment the market. However, as proven by Mussa and Rosen (1978) and more recently by Anderson and Celik (2014), a profit-maximizing firm should always choose inefficient quality but efficient price for its lowest-quality product while doing the reverse for its highest-quality product.[2] Said differently, the low-end product's price should be positive and maximize the single-product profit. This theoretical prescription seems to stand in exact opposite to the notion of freemium.

A number of straightforward reasons come to mind as to why firms find freemium attractive. The illusion of advertising as a "last resort" revenue source is often cited. Saving users the hassle of payment (which can seem high when the price is too small) is another. The power of "free" as a behavioral marketing tool is a third. Although these factors are certainly relevant, we join a nascent literature in marketing and information systems that looks at the more nuanced economic reasons behind the freemium phenomenon. In marketing, two pioneering papers by Kamada and Ory (2015) and Lee et al. (2017) have studied the design of freemium to facilitate word-of-mouth and product diffusion. In comparison, we adopt a single-period monopolistic screening framework and study the optimality of freemium when diffusion-related factors are absent. That is, we ask whether and when "perpetual freemium" remains an effective strategy once a product has achieved sufficient recognition and diffusion-related factors have declined in importance.[3] This is especially relevant for firms that have almost reached market saturation. Google Drive and LinkedIn are best examples in which word of mouth or diffusion is a nonissue.

More specifically, we ask whether network effects from product usage alone can justify the freemium model, when a firm's sole objective is its single-period product line profit. The notion of network effects speaks to the fact that consumers' valuation of a product varies depending on how many other consumers are using the product or compatible products (e.g., Katz and Shapiro 1985, 1986; and Farrell and Klemperer 2007). Network effects can be generated not only by direct interactions of consumers but also by indirect behavioral reasons. Direct interaction happens when a free user of Dropbox shares a file with a paid user, when a free player of Farmville trades with a paid user, and when a free user of Spotify shares her playlist. Behaviorally, network effects are created when a consumer values the product more if there are more users of the same product because it allows him/her to socially fit in with their peers, or when a consumer values the paid product more if there are more users of the free product because he/she can derive social prestige from using the high-end product. Intuitively, offering a free version brings more users on board and generates greater network effects. Meanwhile, the insights of Mussa and Rosen (1978) remain valid, and the risk of cannibalization remains high. Ex ante, it is not clear which is the dominant factor.

By endogenizing a firm's price decision in all relevant subgames, in the baseline model we build a general framework to study a firm's product line strategy. We pay particular attention to whether and when the firm would *endogenously* choose a zero price for its low-end product. Our first set of results speaks to the conditions under which freemium will *not* hold. We show that, as expected, freemium cannot emerge in a classic screening model without network effects. We are able to prove this with a very general quasi-linear utility function and type distribution, thereby showing the robustness of the Mussa and Rosen (1978) insights. Importantly, even with uniform network effects, freemium remains a dominated strategy. Although network effects lead to stronger incentives to expand the market, they also make the cannibalization effect stronger as more users adopt the low-end ("free") product. When the free product delivers the same network value as the paid product, a price cut on the low-end product will increase the attractiveness of both products by the same margin, thus *tightening* each consumer's incentive compatibility constraint. Thus, more free users does not translate into higher profits from the paid users. Consequently, although network effects give the firm stronger incentives to cover the market, this is done by offering a "conventional product line," wherein the low-quality product is priced at a positive level. This result remains valid when we endogenize quality choices for the product line. Put differently, introducing sufficient difference only in "standalone" qualities is not enough to make the freemium strategy viable.

We further show that freemium could indeed become optimal when there is sufficient asymmetry in network effects between the high- and low-end products. For freemium to be viable, the firm's product line has to be such that the paid users gain access to larger network effects compared with their nonpaying peers. This result somewhat echoes the message in Kumar (2014) that to make freemium work, a firm has to offer different sets of features in its free and paid products; but we show that it is the network effects, rather than the "standalone" quality, that are the crucial factors.

As an extension, we endogenize the firm's quality decision of products and examine how the optimal quality levels should change with respect to (w.r.t.) the network effects. To do so, we consider a specific linear utility function and a uniform distribution for consumer type. We show that, in conventional product line design, the optimal quality of the low-end product increases when network effects are higher. Put differently, standalone quality and network effects are complementary to each other. However, in a freemium equilibrium, the low-end product's quality decreases when network effects are larger. In other words, the low-end product's network effect is a substitute for its standalone quality. This result stems from the fact that in a freemium equilibrium the entry-level product generates no revenue, and its own purpose is to expand a firm's user base. The quality provision should be "just enough" to bring enough users on board.

As a second extension, we fully endogenize the firm's product line decisions—quality, price, and network effects—by examining a simpler model in which type distribution is discrete. Remarkably, the main insights from the general model remain intact. We compare the qualities of both products provided under freemium and those offered under conventional product line design. Our analysis suggests that, when the firm adopts the freemium strategy, the (nonnetwork) quality gap between the high-end and the low-end products actually shrinks. When adopting freemium strategy, the firm should even provide a low-end product whose quality is above the efficient level. In other words, the firm should offer higher quality and a zero price to retain the low-type consumers; this surprising result stands in contrast to the results of Mussa and Rosen (1978) of efficient price and inefficient quality. In an optimal product line, quality and network effects are substitutes for the low-end product but are independent dimensions for the high-end product.

To sum up, our analysis yields a set of managerial recommendations that complement what has been suggested in the previous literature. We show that in the absence of word of mouth, "getting more consumers on board" alone cannot justify the freemium strategy. In most cases, market expansion can be more effectively achieved by offering a conventional product line, wherein the low-end product is priced at a positive level to avoid unnecessary cannibalization. In the current framework, perpetual freemium is only optimal under network effects asymmetry. The right freemium strategy should include a free product with lower network benefit than the paid product but superior standalone functionalities (compared with the efficient level).

The difficulty of establishing freemium arising in equilibrium needs special mention. When consumers derive positive utility for a product, the firm can always charge a positive price and such a deviation is inherently likely to be more profitable. Thus, in general, sustaining freemium in equilibrium is likely to be difficult. Given the extensive and growing nature of freemium, however, demonstrating that such a strategy can arise in equilibrium assumes significant importance.

In the markets where freemium is common, externality benefits are also quite common. Therefore, incorporating externality is a natural and arguably critical element of model formulation to examine product line price and qualities in equilibrium. Despite a rich model, we get a sharp insight that asymmetric network externality is essential to support freemium. An equally important sharp insight is that freemium will not be sustained when the network effects are the same across all levels of products that differ in quality.

We organize the rest of the paper as follows. In Section 2 the related literature and the contribution of the present paper are discussed. Section 3 presents the model setup. Section 4 presents the analysis and results. Section 5 discusses the extension on endogenous quality and network effect decisions, as well as the discrete case. Section 6 concludes.

2. Literature Review

This paper is related to three streams of literature. First, a number of recent papers have studied various aspects of the freemium strategy. In marketing, Lee et al. (2017) and Kamada and Ory (2015) are among the first studies on the design of freemium. Kamada and Ory (2015) build a micro model of referral behavior and investigate whether a free contract or a referral program is a more efficient means to encourage word of mouth. The free contract ensures that a receiver would adopt the product even if she turns out to be a low type. When a receiver's adoption generates network effects on the sender, the free contract gives the sender stronger incentives to refer the product in the first place. The main trade-off is between expanded second-period demand (due to word of mouth) and cannibalization. Our model shares some of the features in Kamada and Ory (2015). However, instead of network externalities between first-period senders and second-period receivers, we consider a static model in which network effects exist within and between consumer segments. In our model an expanded network size leads to the potential to increase the high-end product's price, whereas a zero price for the low-end product leads to cannibalization. In other words, the focus of this paper is on the optimality of freemium when diffusion dynamics are absent (Chatterjee and Eliashberg 1990, Mahajan et al. 1990).

In another closely related paper, Lee et al. (2017) develop a structural model to study the design of freemium. Although the paper's focus is empirical, it develops a rich model of consumer behavior that encompasses adoption, upgrade, referral, and usage. There are two main differences between Lee et al. (2017) and this paper. First, this paper considers network effects from product usage but does not model diffusion dynamics. Second, Lee et al. (2017) study the design of freemium once the firm has already committed to a zero price for its low-end product. Even though this is a realistic setup in many contexts wherein a firm would commit to freemium for strategic reasons,[4] we are interested primarily in when and why freemium would endogenously emerge to maximize product line profit. Thus, we endogenize the price on the low-end product instead of fixing it to zero.

A number of papers from information systems have studied various aspects of free trial, popular in the software industry (Cheng and Tang 2010, Dey et al. 2013, Niculescu and Wu 2014, Wagner et al. 2014). None of these papers have fully endogenized price in a general model with a general distribution of consumer type. In particular, we allow the low-end product's price to be endogenous.

Therefore, our study is closely related to the rich literature on product line design in both economics and marketing (e.g., Mussa and Rosen 1978, Moorthy and Png 1992, Varian and Shapiro and 1998, Desai 2001, Desai et al. 2001, Riggins 2003, Orhun 2009, Jing and Zhang 2011, and Anderson and Celik 2014). We follow the paradigm established in Mussa and Rosen (1978) and consider single-period product line profit as the firm's objective function. The firm chooses how many products to offer and sets a price for each product. As shown more recently in Anderson and Celik (2014), without network effects, the optimal product line problem can be reformulated as a multistep optimization problem. The firm first chooses the lowest-quality product's price to maximize its revenue, then proceeds to maximize the additional revenue that comes from the second-lowest-quality product. Although the standard Mussa and Rosen model does not consider network effects, a number of recent papers have examined the impact of network effects. Jing (2007) examined market segmentation under network externalities and found that the existence of network effects gives the firm stronger incentives to cover the market. The author did not explore the case of freemium, but his main insights are echoed in this paper.

Finally, in a broad sense, an asymmetric product line is somewhat reminiscent of a two-sided market. In a two-sided market setup, a platform (firm) has incentives to lower the price for one side below the marginal cost as long as this brings value to the other side (e.g., Hagiu 2006 and Rochet and Tirole 2006). Although cannibalization is not relevant in a two-sided market context, it is in the context of product line design. We show that it is the coupling of network effects and cannibalization effect that makes the freemium problem unique.

3. Model

In analyzing the freemium problem, we intend to make our key insights as general as possible and independent of most specific assumptions on functional form. Thus, we start by presenting a general model in which we make few assumptions on the form of the utility function as well as consumer type distribution. At the same time, we present a running example with linear utility function and uniform distribution of consumer type. This allows us to precisely pin down the conditions for freemium in analytical forms, and we hope that this exercise will strengthen our main intuitions. Next we present the general model and the running example in turn.

Consider a monopolist who has the option of offering either one or two vertically differentiated products. For notational convenience, we denote the firm's product strategy as Ω. If two products are offered, $\Omega = \{L, H\}$, where L, H stands for the products of relatively low and high quality. If only one product is offered, Ω is a singleton.

There is a unit mass of consumers. They have heterogeneous taste, which is described by the distribution of θ, a density $f(\theta)$ defined on $[0,1]$. All customers have access to an outside option, the utility of which is u_0. In the case of Dropbox, for example, this outside option denotes the utility a consumer gets from using a traditional form of storage.[5]

For a customer with taste parameter θ, her valuation from consuming product i is $V^i(\theta, D)$, where $i \in \Omega$ and D is the total user base of the firm's product. Let $D = D^{-i} + D^i$, where D^i is the demand for product i, and D^{-i} is the demand for the other type of product if offered. In this framework, $\frac{\partial V^i(\theta,D)}{\partial D}$ captures network benefit for consumers using product i. In other words, we consider a type of "global" network effects whereby only the total network size affects consumer utility, though the relationship does not have to be linear. The total utility a consumer derives from buying product i is therefore $U^i = V^i(\theta, D) - p^i$, where p^i is the price of product i.

We make the following assumptions regarding $V^i(\theta, D)$.

AI. Strictly increasing in quality. $\forall \theta, D,$

$$V^H(\theta, D) > V^L(\theta, D).$$

AII. Differentiable in D, *and* strictly increasing in D. $\forall \theta, i,$

$$\frac{\partial V^i(\theta, D)}{\partial D} > 0.$$

AIII. Differentiable in θ, *and* strictly increasing in θ. $\forall D, i,$

$$\frac{\partial V^i(\theta, D)}{\partial \theta} > 0.$$

AIV. Has increasing differences in θ and quality/network effects. $\forall D,$

$$\frac{\partial [V^H(\theta, D) - V^L(\theta, D)]}{\partial \theta} > 0.$$

Assumption AIV can be alternatively and more restrictively stated as two assumptions that, respectively, speak to the increasing differences conditions regarding consumer type and the standalone product quality/network effects, that is, $\frac{\partial [V^H(\theta,0) - V^L(\theta,0)]}{\partial \theta} > 0$ and $\forall D$, $\frac{\partial [V^H(\theta,D) - V^L(\theta,D)]}{\partial D \partial \theta} \geq 0$.[6] Here, $V^i(\theta, 0)$ captures the consumer's valuation of product i's quality when consumers do not derive utility from other users (equivalently $D = 0$), and $\frac{\partial V^i(\theta,D)}{\partial D}$ captures the marginal network effect derived from using product i, which has user base D. The first inequality implies $V^i(\theta, 0)$ has increasing difference in (θ, i), whereas the second inequality means the network benefit also has increasing difference in (θ, i). Assumption AIV relaxes the conditions and requires only $V^i(\theta, D)$ has increasing difference in (θ, i).

The game consists of two stages. First, the firm chooses its menu of products. Next, consumers make purchase decisions conditional upon the firm's menu and belief about all other consumers' decisions. As is typical in a game with network effects, multiple equilibria may exist in the second stage. Specifically, we seek the Nash equilibrium in the second stage game that is Pareto dominant. Assumption AIII guarantees that a consumer of type θ_0 would expect that all other consumers with $\theta > \theta_0$ have a higher evaluation for any product. Therefore, if type θ_0 prefers purchasing product i to the outside option, type $\theta > \theta_0$ would also do so. Similarly, if type θ_0 prefers purchasing the higher-quality product to the lower-quality one, he would expect type $\theta > \theta_0$ to do the same. In Appendix A, we show that in the baseline model, the Pareto dominant outcome consists of one of two outcomes. In the first scenario, the firm offers the high-quality product only. In the second scenario, the firm offers a product line, with the higher-valuation segment buying the high-quality product and the lower-valuation segment buying the low-quality product.

The assumptions made above are consistent with those in classic papers on product line design, except we also account for network effects. Lemma 1 illustrates how the demand schedules are determined for each product and pricing strategy.

Lemma 1. *Assume that both types of products are offered, and the prices are such that both have positive sales. The demand schedule is determined by two marginal consumers at locations θ_L and θ_{HL}, where*

$$V^L(\theta_L, D) - p^L = u_0, \quad (1)$$
$$V^H(\theta_{HL}, D) - p^H = V^L(\theta_{HL}, D) - p^L \quad (2)$$

with the low-end product serving $[\theta_L, \theta_{HL}]$, and the high-end product serving $[\theta_{HL}, 1]$. When only the high-end product is offered, the marginal customer type θ_H is determined by

$$V^H(\theta_H, D) - p^H = u_0 \quad (3)$$

and only consumers with $[\theta_H, 1]$ are served.

The proof for Lemma 1 is as follows. In choosing between two types of products, a consumer of type θ chooses the low-end product only if $p^H - p^L > V^H(\theta, D) - V^L(\theta, D)$. Given Assumption AIV, the larger θ is, the greater the reduction in price is required for a consumer to choose the low-end product. Hence, it is impossible to induce a consumer of type θ_i to purchase a low-end item if the high-end product is purchased by a consumer of type $\theta_j < \theta_i$. From this feature of $V^i(\theta, D)$ and from the fact that the monopolist can make positive profits from serving at least the high-θ consumers, it follows that if both types of products are offered, the monopolist serves $[\theta_L, \theta_{HL}]$ with the low-quality product and $[\theta_{HL}, 1]$ with the high-quality product, where θ_L denotes the marginal consumer who is indifferent between purchasing the low-end product and the outside option, whereas θ_{HL} denotes the marginal consumer who is indifferent between purchasing the high-end and low-end product. The firm's profit is thus $\Pi_{HL} = D^H p^H + D^L p^L$. In the case where only the high-end product is offered,[7] the firm serves $[\theta_H, 1]$, where θ_H denotes the marginal customer who is indifferent between purchasing the high-end product and the outside option. In this case, the firm's profit is simply $\Pi_H = D^H p^H$.

A freemium equilibrium is one in which the firm offers both types of products but charges a zero price for the low-end product. It is formally defined as follows:

Definition 1. A freemium equilibrium is defined as a product line offering both products H and L, wherein $p^{L*} = 0$, $p^{H*} > 0$, $\Pi^*_{HL} > \Pi^*_H$.

To explain our findings more clearly, throughout the paper we will illustrate our general findings with a running example with valuation function $V^i(\theta, D) = \theta q^i + \alpha^i D$ and uniform distribution of consumer types (i.e., $\theta \sim U[0,1]$). The quality of product i is denoted by q^i, and α^i captures the network benefit one can derive from any other user's usage of product i. This running example satisfies all Assumptions AI–AIV.

In this running example, the demand for each product and the total demand are given by

$$D^H = \int_{\theta_{HL}}^{1} f(\theta)\, d\theta = 1 - \theta_{HL},$$
$$D^L = \int_{\theta_L}^{\theta_{HL}} f(\theta)\, d\theta = \theta_{HL} - \theta_L,$$
$$D = D^H + D^L = 1 - \theta_L,$$

where the marginal consumers θ_L, θ_{HL} are given by

$$\theta_L q^L + \alpha^L D - p^L = u_0,$$
$$\theta_{HL} q^H + \alpha^H D - p^H = \theta_{HL} q^L + \alpha^L D - p^L.$$

The total profit for the monopolist is thus

$$\Pi_{HL} = p^H D^H + p^L D^L.$$

In the baseline analysis, we therefore endogenize the product set choice as well as the pricing decision. We do not, however, endogenize the quality level or the level of network effects. In extensions, we endogenize both decisions by considering model formulations that are analytically tractable. According to Definition 1, freemium is different from selecting a price equal to the marginal cost. As demonstrated in extensions (Section 5) where quality decision is endogenized, incorporating a positive production cost does not qualitatively affect our results in the main analysis (Section 4).

Before we proceed with the analysis, let us briefly discuss our formulation of network effects. We choose a relatively simple formulation wherein each product delivers a different level of network effect (e.g., α_L and α_H in the running example). This is a somewhat standard formulation whereby network effect is considered as a product attribute. In reality, however, the patterns of network effects can be much more complex. For example, in social games, the network effects are governed by the consumers' obtained utilities when they interact with each other in games. The nonpaying users may play at a disadvantage and derive a disutility when interacting with paying users. Though this scenario will indeed imply that the free option generates less overall network effects than the paid option, the exact level of disutility versus utility would depend on the frequency of interaction between the two types of users. In other words, it can be best captured by a case of *local* network effects with four, instead of two, parameters. Throughout the analysis, we choose to present the simplest model where the network effects can be parameterized by two parameters. It should be kept in mind that our notion of asymmetric network effects can be richer than it seems, capturing cases such as the social game example discussed above. A more detailed analysis that covers the case of quality-dependent and local network effects is in Online Appendix A.

4. Analysis

The game has two stages: first, the firm decides whether to offer one or two products and sets the price for the offered product(s); second, consumers decide on whether to purchase the product that gives the highest utility or purchase nothing. To explore the conditions under which freemium is an optimal strategy, we discuss three cases in turn:

1. No network effect $\frac{\partial V^i(\theta,D)}{\partial D} = 0$ for all $i \in \{L,H\}$.
2. Uniform (symmetric) network effect $\frac{\partial V^i(\theta,D)}{\partial D} = \alpha > 0$ for all $i \in \{L,H\}$, where α is a constant.
3. Asymmetric network effect $\frac{\partial V^i(\theta,D)}{\partial D}$ differs for $i = L$ and $i = H$.

To prove the optimality of freemium, we consider a necessary condition for freemium: $\frac{\partial \Pi}{\partial p^L}|_{p^L=0} < 0$. In words, the monopolist would like to set a zero price for the low-end product, only when he has no incentives to marginally increase the price when it is already at zero. It turns out that this condition is sufficient to rule out the optimality of freemium in cases (1) and (2), under the general functional form. Let θ_L, θ_H and θ_{HL} be defined by Equations (1), (2), (3) and the asterisk $*$ refer to the equilibrium value. Assuming uniform or no network effects, Proposition 1 states the nonexistence of freemium under uniform network effects, and Corollary 1 illustrates it with the running example. When $\frac{\partial V^i(\theta,D)}{\partial D} = \alpha \geq 0$ for $i \in \{L,H\}$, freemium can never emerge as the equilibrium strategy.

Proposition 1. *When $\frac{\partial V^i(\theta,D)}{\partial D} = \alpha \geq 0$ for $i \in \{L,H\}$, freemium can never emerge as the equilibrium strategy.*

For the running example, Proposition 1 can be stated in a more explicit way as in Corollary 1.

Corollary 1. *With $\alpha^i = \alpha \geq 0$ for $i \in \{L,H\}$, $V^i(\theta,D) = \theta q^i + \alpha D$ and $\theta \in U[0,1]$, the firm's equilibrium product and pricing strategy can be characterized as follows:*

a. *When $q^L > \alpha > u_0$ and $q^L + u_0 \geq 2\alpha$, the firm offers both products with $p^{L*} = \frac{q^L-u_0}{2}$, $p^{H*} = \frac{q^H-u_0}{2}$, $\theta^*_{HL} = \frac{1}{2}$, $\theta^*_L = \frac{q^L+u_0-2\alpha}{2(q^L-\alpha)}$ and $\Pi^*_{HL} = \frac{q^L-u_0}{4}\left(\frac{\alpha-u_0}{q^L-\alpha}\right) + \frac{q^H-u_0}{4}$.*

b. *Otherwise, the firm offers only the high-end product, with $p^{H*} = \frac{q^H-u_0}{2}$, $\theta^*_H = \frac{u_0+q^H-2\alpha}{2(q^H-\alpha)}$ and $\Pi^*_H = \frac{(q^H-u_0)^2}{4(q^H-\alpha)}$.*

We explain the intuition behind Proposition 1 with two subcases: one without network effects (i.e., $\frac{\partial V^i(\theta,D)}{\partial D} = 0$ for all $i \in \{L,H\}$), and the other with uniform network effects (i.e., $\frac{\partial V^i(\theta,D)}{\partial D} = \alpha > 0$ for $i \in \{L,H\}$). Although the former is a special case of the latter, separating them helps us build some intuitions. The intuition for the case with zero network effects is consistent with what has been shown in the product line design literature and has been most succinctly summarized by Anderson and Celik (2014). In a nutshell, setting the low-end product's price to zero generates no revenue *and* puts downward pressure on the high-quality product's price and demand. Thus, the firm can be better off by withdrawing the low-end product altogether. When it does launch the entry-level product, it always sets a price that is efficient from a single product profit maximization standpoint (see Anderson and Celik 2014 for details).

When network effects are present, how would the firm's optimal product strategy be impacted? More specifically, can network effects lead to an equilibrium wherein the firm pursues the freemium strategy? As discussed in the introduction, increasing network size is a major intuition in favor of the freemium strategy. Although the low-end product generates no revenue and partially cannibalizes the high-end sales, a larger network size brings higher utility to the high-valuation customers, possibly leading to higher price and, therefore, profit. This is akin to the strategy of user subsidization in a two-sided market context. However, a formal analysis reveals that this intuition is not valid when the network benefits for users of both high- and low-end products are positive but identical.

What is the intuition behind? For freemium to be optimal, the optimal price for the low-end product must be zero. A necessary condition for this is $\frac{\partial \Pi_{HL}}{\partial p^L}|_{p^L=0} \leq 0$. In other words, the firm has incentive to decrease the low-end price even if it is already at or close to zero. When network effects are symmetric or not too different between the two products, this condition cannot be met. At $p^L = 0$, it is straightforward that $\frac{\partial \Pi_L}{\partial p^L}|_{p^L=0} \geq 0$. Thus, $\frac{\partial \Pi_{HL}}{\partial p^L}|_{p^L=0} \leq 0$ requires that $\frac{\partial \Pi_H}{\partial p^L}|_{p^L=0} \geq 0$. In other words, at an infinitesimal p^L, the profit from the high-end product would increase as the firm decreases its price on the low-end product. A necessary condition for this is that the marginal consumer's incentive compatibility constraint is relaxed as p^L is reduced. This is necessary for either the demand or the price of the high-end product to increase. However, under uniform network effect, this is not possible for the following intuition. As the firm lowers the price for its low-end product, the total user base expands. Because of network effects, each user indeed gains greater surplus from using the high-end product. However, because network effects are symmetric, the low-end product also becomes more attractive, by the same margin. In other words, each consumer's incentive compatibility constraint has not been relaxed. The marginal consumer, in fact, now prefers the low-end product more because of a lower price. Thus, the firm cannot charge a higher p^H, nor will it have a higher demand from the high-end product. Meanwhile, the firm suffers greater loss from the low valuation segment. Taken together, in the subgame wherein the firm offers two products, decreasing p^L while it is close to zero always decreases firm profit. Please see Appendix A for the formal proof.

In practice, it is certainly rare that different products would deliver exactly the same level of network effects. However, the broad insights remain intact as long as the network benefits of different products are close enough. This corresponds to a wide range of applications for which the firm does not or cannot restrict interaction between paid and free users. In most mobile messaging tools, for example, users can send messages to each other regardless of whether they are paying. The network aspect of the product is a relatively simple and straightforward feature. It is difficult to restrict the network benefits enjoyed by the nonpaying users unless the firm intentionally handicaps the product.

For the case of uniform network effect, Proposition 1 uncovers a fundamental tension between expanding the network size and containing cannibalization. Next, we consider the case in which the firm's high-end and low-end products can deliver different levels of network effects. This is a widely observed practice among firms who successfully pursue the freemium strategy. LinkedIn, for example, gives free users only limited access to view others' profiles, especially contact details. In some of its freemium games, Zynga used to charge "entry tickets" if the users wanted to game with other users. Paid users of Dropbox are able to share more files with others than free users. Proposition 2 states that freemium may indeed emerge as an equilibrium strategy when the high-end and low-end products differ on both the baseline quality as well as the network effects dimension.

Proposition 2. *When the following necessary condition is satisfied, freemium can be an equilibrium strategy:*

$$\left[\frac{\partial V^H(\theta_{HL},D)}{\partial D}-\frac{\partial V^L(\theta_{HL},D)}{\partial D}\right]\Big|_{p^{L*}=0,p^{H*}} \geq \left[\frac{1+\frac{\int_{\theta_L}^{\theta_{HL}} f(\theta)d\theta}{p^H f(\theta_{HL})}\left(\frac{\partial V^H(\theta_{HL},D)}{\partial\theta_{HL}}-\frac{\partial V^L(\theta_{HL},D)}{\partial\theta_{HL}}\right)}{f(\theta_L)\frac{\partial\theta_L}{\partial p^L}}\right]\Big|_{p^{L*}=0,p^{H*}}.$$

Corollary 2 explicitly states the necessary condition for freemium to be optimal for the running example.

Corollary 2. *With $\alpha^H \neq \alpha^L$, $V^i(\theta,D)=\theta q^i+\alpha^i D$ for $i\in\{L,H\}$, and $\theta\in U[0,1]$, freemium can be the optimal equilibrium strategy when the following necessary condition is satisfied:*

$$\alpha^H-\alpha^L \geq (q^L-\alpha^L)\left[1+\frac{\theta^*_{HL}-\theta^*_L}{p^{H*}}(q^H-q^L)\right]$$

where $0<\theta^*_L<\theta^*_{HL}<1$ *and*

$$\theta^*_L=\frac{u_0-\alpha^L}{q^L-\alpha^L},$$

$$\theta^*_{HL}=\frac{(q^H-q^L)q^L+\alpha^H(u_0-q^L)-\alpha^L(q^H-2q^L+u_0)}{2(q^H-q^L)(q^L-\alpha^L)},$$

$$p^{H*}=\frac{q^L(\alpha^H-q^L)-(\alpha^H-\alpha^L)u_0}{2(q^L-\alpha^L)}+\frac{q^H}{2}.$$

From Proposition 2 and Corollary 2, we can see that freemium can be optimal only if the network effect differential $\alpha^H-\alpha^L$ is higher than some positive value.

To solve the profit maximization problem, when two products are offered, first we can get the prices (p^H, p^L) expressed by marginal consumer types (θ_{HL}, θ_L) by Lemma 1. Substituting the price function back to the profit function Π_{HL}, the optimal θ^*_L and θ^*_{HL} are the values that maximize Π_{HL} under the constraint $0\leq\theta_L<\theta_{HL}<1$ and $p^L\geq 0$, and then the corresponding p^{L*} and p^{H*} can be obtained. When only the high-end product is offered, we can solve for θ^*_H and thus p^{H*} following similar logic. For the running example, the profit function is concave in θ_k, with $k\in\{H,L,HL\}$; the optimal solutions, thus, can be obtained straightforwardly. The following Corollary 3 gives the optimal product and pricing strategies and therefore describes a sufficient condition for freemium to be optimal.

Corollary 3. *With $\alpha^H\neq\alpha^L$, $V^i(\theta,D)=\theta q^i+\alpha^i D$ for $i\in\{L,H\}$, and $\theta\in U[0,1]$, the firm's equilibrium product and pricing strategy can be characterized as follows:*

a. *When* $\frac{\partial\Pi_{HL}}{\partial p^L}\big|_{p^{L*}=0,p^{H*}}\leq 0$, $0<\frac{u_0-\alpha^L}{q^L-\alpha^L}<\frac{(q^H-q^L)q^L-\alpha^L(q^H-2q^L+u_0)+\alpha^H(u_0-q^L)}{2(q^H-q^L)(q^L-\alpha^L)}<1$, *and* $\Pi^*_{HL}\geq\frac{(q^H-u_0)^2}{4(q^H-\alpha^H)}$, *the firm offers two products with*

$$p^{L*}=0,$$

$$p^{H*}=\frac{q^L(\alpha^H-q^L)-(\alpha^H-\alpha^L)u_0}{2(q^L-\alpha^L)}+\frac{q^H}{2},$$

$$\theta^*_L=\frac{u_0-\alpha^L}{q^L-\alpha^L},$$

$$\theta^*_{HL}=\frac{(q^H-q^L)q^L-\alpha^L(q^H-2q^L+u_0)+\alpha^H(u_0-q^L)}{2(q^H-q^L)(q^L-\alpha^L)},$$

$$\Pi^*_{HL}=\frac{[\alpha^L(q^H-u_0)-q^L(q^H-q^L)+\alpha^H(u_0-q^L)]^2}{4(q^H-q^L)(q^L-\alpha^L)^2}.$$

b. *When* $0 \le \frac{p^{L*}+u_0-\alpha^L}{q^L-\alpha^L} < \frac{p^{H*}-p^{L*}-(\alpha^H-\alpha^L)\frac{q^L-p^{L*}-u_0}{q^L-\alpha^L}}{q^H-q^L} < 1$, $0 < p^{L*} < p^{H*}$, *and* $\Pi^*_{HL} \ge \frac{(q^H-u_0)^2}{4(q^H-\alpha^H)}$, *the firm offers both products with positive prices and*

$$\theta^*_L = \frac{(\alpha^H-\alpha^L)(\alpha^H-\alpha^L-q^H+q^L) + 2(q^H-q^L)(\alpha^H+\alpha^L-u_0-q^L)}{(\alpha^H-\alpha^L)^2-4(q^H-q^L)(q^L-\alpha^L)},$$

$$\theta^*_{HL} = \frac{(\alpha^H-\alpha^L)^2+\alpha^L(u_0+2q^H-3q^L) + \alpha^H(q^L-u_0)-2q^L(q^H-q^L)}{(\alpha^H-\alpha^L)^2-4(q^H-q^L)(q^L-\alpha^L)},$$

$$\Pi^*_{HL} = \frac{(q^H-q^L)[\alpha^L(q^H-c)-\alpha^H(q^L-c) + (2c-q^H)q^L-c^2]}{(\alpha^H-\alpha^L)+4(\alpha^L-q^L)(q^H-q^L)}.$$

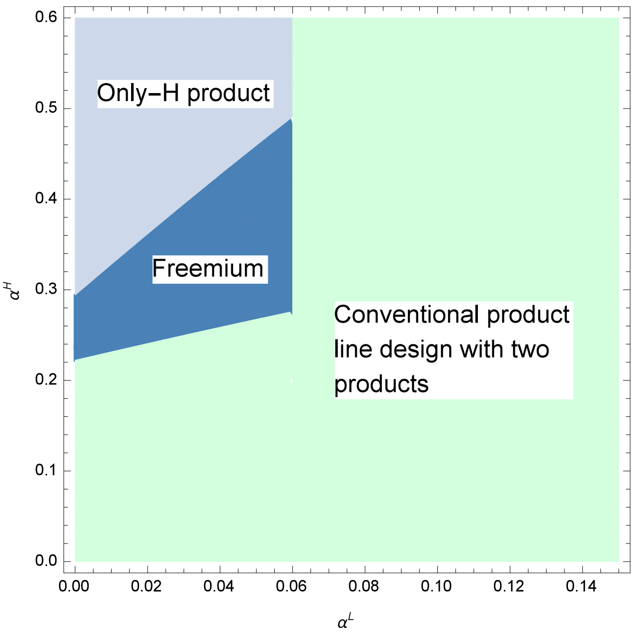
c. *Otherwise the firm offers only the high-end product with* $p^{H*} = \frac{q^H-u_0}{2}$, $\theta^*_H = \frac{q^H+u_0-2\alpha^H}{2(q^H-\alpha^H)}$, $\Pi^*_H = \frac{(q^H-u_0)^2}{4(q^H-\alpha^H)}$.

Under asymmetric network effects (i.e., $\frac{\partial V^H(\theta_{HL},D)}{\partial D} - \frac{\partial V^L(\theta_{HL},D)}{\partial D}$ has to be greater than a certain positive threshold), the firm can indeed increase total profit by offering the low-end product for free. The key intuition is as follows. As the firm cuts its low-end price to expand demand, both high- and low-end products become more attractive. However, because of the difference in network effects $\frac{\partial V^H(\theta_{HL},D)}{\partial D}$ and $\frac{\partial V^L(\theta_{HL},D)}{\partial D}$, the high-end product becomes relatively more attractive. Put differently, the incentive compatibility constraint for the marginal consumer can be less tight when the network size is larger, owing to asymmetric network effects. As such, when $\frac{\partial V^H(\theta_{HL},D)}{\partial D} - \frac{\partial V^L(\theta_{HL},D)}{\partial D}$ is large enough, holding p^H fixed, decreasing p^L may lead to higher demand for the high-end product and therefore higher profit. The premium from the paid users is indeed sufficient to pay for the losses.

We can also see that in the running example, with asymmetric network effects, $q^H - q^L$ needs to be small enough. Intuitively, if $q^H - q^L$ is too large, even with $p^L = 0$, the low-end product quality q^L is not high enough to attract many new customers (i.e., θ_L is too large) and is thus unable to create high enough network benefit. In this case, even a fairly large asymmetry between α^H and α^L cannot induce freemium as the optimal strategy (the necessary condition cannot be satisfied). This condition makes it sufficient for freemium to be adopted as an equilibrium strategy. Please see Appendix A for a more detailed discussion.

Figure 1 illustrates these equilibrium strategies in the parameter space for the running example. Simply put, when the difference between the network effects of low- and high-end products is relatively large, the firm has less pressure from cannibalization and focuses on

Figure 1. (Color online) Product Line Strategy Under Network Effects ($q^H = 0.8$, $q^L = 0.2$, $u_0 = 0.06$)



network effects. This leads to the freemium strategy, wherein the firm forgoes the profit from the low-end customers and considers them a subsidy to the premium users. When the network effects of the low-end product become stronger, the firm pursues a conventional product line strategy, whereby prices for both products are positive. When the high-end product's network effects (i.e., α^H) are very high, the firm would be better off by offering only the high-end product.

It is worth noting that, as shown in Figure 1, if we decrease α^L while fixing α^H, the optimal strategy may move from freemium to offering only the high-end product. This may seem puzzling at first glance because a lower α^L leads to a larger asymmetry between the network effects, which further relaxes the cannibalization pressure. However, another effect of lowering α^L is that the firm would find it harder (costlier) to attract consumers to use the low-end product, and therefore offering only the high-end product becomes more attractive. Mathematically, as shown by the necessary condition in Proposition 2 and Corollary 2, when α^L decreases Δ, the left-hand side of the inequality increases Δ, but the right-hand side increases more than Δ, making it harder to satisfy the inequality. Similarly, if we increase α^H while fixing α^L, the optimal strategy may also move from freemium to offering only the high-end product. This is because that the profit of offering only the high-end product increases in α^H. Therefore, even though the necessary condition for freemium to be optimal is easier to satisfy with a larger α^H, the sufficient condition, which requires $\Pi^*_{HL}|_{p^{L*}=0, p^{H*}} > \Pi^*_H$, is harder to satisfy.

Table 1. Summary of Product and Pricing Choices in Equilibria

Type of equilibrium	Features	Exists under no network effect?	Exists under uniform network effect?	Exists under asymmetric network effect?
(1) Conventional product line	Anderson and Celik (2014), $p^{H*} > p^{L*} > 0$	✓	✓	✓
(2) Freemium	$p^{L*} = 0$, $p^{H*} > 0$	×	×	✓
(3) Only one product	$p^* > 0$	✓	✓	✓

Table 1 provides a summary of our main insights. We describe the product line choices under each type of equilibrium and provide their existence conditions.

It should be noted that the phenomenon of freemium can be considered as a special case of complementary good pricing. A monopoly that sells two complementary goods has incentives to lower each product's price if this leads to higher sales of the other product. If the complementarity by one product to the other product is stronger, the firm may lower the former product's price below marginal cost in order to profit from the sales of the latter product. Our model essentially builds on this insight in a vertical differentiation scenario. We argue that in a vertical product line, the cannibalization effect coexists with the complementarity effect. In the case of network effects, the network effects have to be asymmetric so that the complementarity effect outweighs the cannibalization effect when the firm lowers the entry-level product's price. The economic intuition, however, may indeed have broader applications.[8]

5. Endogenous Quality Decisions

The analyses thus far speak to the conditions under which freemium is or is not optimal in a product line with given qualities and network effects. In this section we consider the endogeneous determination of the quality levels and network effects. This is a technically challenging exercise, and we approach it by considering two alternative formulations. Section 5.1 considers the running example introduced in Section 3, with uniform distribution of consumer type and linear utility function. We allow q^i to be a decision variable and let $C(q^i)$ denote the firm's marginal production cost of products with quality q^i. This formulation allows us to study the optimal determination of standalone qualities (q^i) when network effects are given. In Section 5.2 we fully endogenize both q^i and α^i in a discrete segment model, which corresponds to the widely used model of two-type screening.

Remarkably, the key insights in Section 4 hold true in all the alternative formulations. Thus, this section could also be considered as a robustness check, while we make more variables endogeneous in progressively simpler models. In both subsections, we relegate the proofs to Appendix A and rely on numerical methods to generate the graphical illustrations.

5.1. Endogenous Quality

With q^H, q^L, p^H, p^L all endogenized, we take a closer look at the equilibrium levels of q^i as a function of network effects. That is, how should the product qualities shift with network effects?

To proceed, we need the following two assumptions in addition to AI–AIV.

AV. $V^i(\theta, D)$ is concave in q^i, $\forall i \in \{H, L\}$.

AVI. $C(q^i)$ is convex in q^i, $\forall i \in \{H, L\}$.

With the above two assumptions together with AI–AIV, we find that the firm should respond to higher α^L by increasing q^L in the conventional product line. However, it should respond to higher α^L by reducing quality provision q^L while adopting freemium. Proposition 3 illustrates the firm's choices of quality in the conventional product line and freemium regime, with the running example satisfying AI–AIV and marginal cost satisfying AVI.

Proposition 3. *With valuation function $V^i(\theta, D) = \theta q^i + \alpha^i D$, marginal production cost $C(q^i) = c \cdot (q^i)^2$ and $\theta \sim U[0,1]$, when conventional product line is the optimal strategy, the firm offers both products with $q^{H*} = \frac{\theta^*_{HL}}{2c}$, $q^{L*} = \frac{\theta^*_L + \theta^*_{HL} - 1}{2c}$, and q^{L*} increases in α^L; when freemium is the optimal strategy, the firm offers both products with $q^{H*} = \frac{\theta^*_{HLf}}{2c}$, $q^{L*} = \frac{u_0 - \alpha^L(1-\theta^*_{Lf})}{\theta^*_{Lf}}$, and q^{L*} decreases in α^L.*

As explained previously, when a firm pursues the freemium strategy, it suffers a loss on the low-valuation segment and recuperates the lost profit from the high-valuation segment. Because the firm desires the low-valuation consumers for the sole purpose of enlarging its network size, it should always supply the least level of quality that is sufficient to induce purchase. As α^L increases, the minimal required quality level decreases accordingly, leading to lower quality provision. Broadly speaking, quality and network effects are substitutes for the low-end product. The firm always seeks the least costly way to attract the low-end customers, and as one dimension gets higher, it decreases its investment in the other dimension.

5.2. Endogenous Quality and Network Effect

In the baseline model we examined a general utility functional form as well as a general consumer distribution. In Section 5.1 the product quality is endogenized. Next we endogenize both quality and network effect decisions. In reality, the network effect—as a product attribute—may also be the firm's endogenous decision. For example, in social games, the game designer (the firm) can endogenously decide how much network effect the paid users and free users can get, by designing the frequency of interaction between different types of players. In the case of Dropbox, the firm can make sharing more or less convenient so that different products deliver different network effect.

To keep the analysis tractable, we consider a discrete distribution of consumers on the demand side. Namely, there are two segments of consumers, with high and low valuation for the product quality as well as network benefit. Each consumer is characterized by a taste parameter $\theta \in \{\theta_H, \theta_L\}$, where $\theta_H > \theta_L$. A fraction λ of consumers belong to type θ_H, who have higher valuation for the firm's products. A fraction $1-\lambda$ of consumers belong to type θ_L. There may be some debate on the formulation of the running example regarding whether the taste parameter (i.e., θ) affects only the valuation of the standalone quality or the valuation of both quality and network benefit. In short, both formulations satisfy AI–AIV; therefore, the results derived for the general model in Section 4 hold for both specifications. To demonstrate our results hold for both cases, we therefore offer a formulation in this extension different from that used in the previous running example. For a customer with taste parameter θ, her valuation from consuming product i is:

$$V^i(\theta, \alpha^i, D) = \theta(q^i + \alpha^i D),$$

where $D \le 1$ is the total number of users who buy from the firm's product line, namely $D = \sum_{i\in\{H,L\}} D^i$. As such, we assume that each user derives network effects from all other users in the firm's network. The total magnitude of network effects depends on the network size and product design. The firm sets price $p^i > 0$ for each product i. To guarantee the existence of interior solutions, we assume that the marginal cost of serving a consumer is increasing in both q^i and α^i quadratically, that is, $C(q^i, \alpha^i) = c(q^i)^2 + s(\alpha^i)^2$. The firm's product line profit is thus:

$$\Pi = \sum_{i\in\{H,L\}} D^i[p^i - c(q^i)^2 - s(\alpha^i)^2].$$

We can see that all of AI–AIV are satisfied. In this discrete case we also assume that the high-type consumers have positive valuation for the low-end product at price zero.[9] Proposition 4 spells out the optimal product line design under uniform network effect.

Proposition 4. *When $\alpha^H = \alpha^L = \alpha$, the equilibrium product-line strategy can be characterized as follows:*

$$\text{When (I)} \begin{cases} \frac{\theta_L^2 - \lambda^3\theta_H^2}{4s} - (1-\lambda)u_0 > 0 \\ \lambda\theta_H\theta_L > \theta_L^2 \ge 2su \end{cases}$$

or (II)

$$\begin{cases} \frac{(\lambda\theta_H - \theta_L)^2}{4c(1-\lambda)} + \frac{2\theta_L - \lambda^3\theta_H^2}{4s} - (1-\lambda)u_0 > 0 \\ s[\theta_H\theta_L\lambda + 2cu_0(1-\lambda)] \le \theta_L^2(c - c\lambda + s) \\ \lambda \le \frac{\theta_L}{\theta_H} \end{cases}$$

the firm offers both products with price $p^{H*} > p^{L*} > 0$.[10] *The corresponding optimal qualities, network intensity, and firm profit are as follows:*

$$q^{L*} = \begin{cases} 0 & \text{, when (I) holds} \\ \frac{\theta_L - \theta_H\lambda}{2c(1-\lambda)} & \text{, when (II) holds} \end{cases},$$

$$q^{H*} = \frac{\theta_H}{2c},\ \alpha^* = \frac{\theta_L}{2s},$$

$$\Pi^*_{HL} = \begin{cases} \frac{\lambda\theta_H^2}{4c} + \frac{\theta_L^2}{4s} - u_0 & \text{, when (I) holds} \\ \frac{\theta_L^2 + \lambda\theta_H^2 - 2\lambda\theta_H\theta_L}{4c(1-\lambda)} + \frac{\theta_L^2}{4s} - u_0 & \text{, when (II) holds} \end{cases}.$$

Otherwise, the firm offers only a high-end product. The price, quality, and equilibrium profit are:

$$p^* = \frac{\theta_H^2}{2c} + \frac{\theta_H^2\lambda^2}{2s} - u_0,$$

$$q^* = \frac{\theta_H}{2c}, \alpha^* = \frac{\lambda\theta_H}{2s},$$

$$\Pi^*_H = \lambda\left(\frac{\theta_H^2}{4c} + \frac{\lambda^2\theta_H^2}{4s} - u_0\right).$$

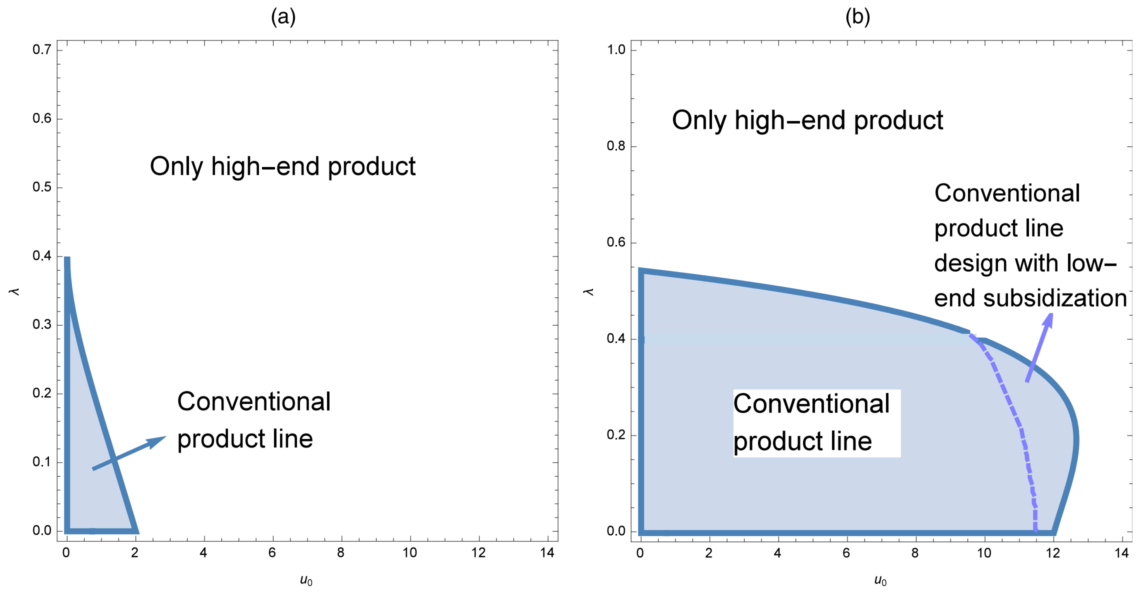
Figure 2 shows regions for optimal product line strategies under zero network effects as well as uniform network effects. Figure 2(a) corresponds to the case in which $\alpha = 0$ (i.e., consumers derive no network benefit from using the product). Figure 2(b) depicts the regions for optimal strategies for the case with $\alpha > 0$, namely, the network effects are positive and symmetric.

From Figure 2, we can see that freemium is never an optimal strategy when network effects are zero or symmetric, consistent with the results of our baseline model.

Next we consider the case in which the firm can design its high-end and low-end products to deliver different levels of network effects. Proposition 5 states the optimal product line design when the network effects can be designed as asymmetric.

Proposition 5. *When the firm can set α^H and α^L at different levels, the equilibrium product-line strategy can be characterized as follows:*

Figure 2. (Color online) Regions for Optimal Strategies ($c = 0.5, s = 0.1, \theta_H = 5, \theta_L = 2$)



Note. (a) Case without network effect; (b) case with uniform network effect.

When $\frac{\theta_H^2}{4s}(\lambda - \lambda^3) + \lambda u_0(1 - \frac{\theta_H}{\theta_L}) - \frac{csu_0^2(1-\lambda)}{\theta_L^2(c+s)} \geq 0$ *and* $\frac{2csu_0(1-\lambda)}{\theta_L^2(c+s)} + \frac{\lambda\theta_H}{\theta_L} - 1 > 0$, *the firm adopts freemium strategy with:*

$$p^{L*} = 0, \ p^{H*} = \frac{\theta_H^2}{2}\left(\frac{1}{s} + \frac{1}{c}\right) - \frac{u_0\theta_H}{\theta_L}.$$

The corresponding optimal qualities, network effects, and profit are as follows:

$$q^{L*} = \frac{su_0}{\theta_L(c+s)}, \ q^{H*} = \frac{\theta_H}{2c},$$
$$\alpha^{L*} = \frac{cu_0}{\theta_L(c+s)}, \ \alpha^{H*} = \frac{\theta_H}{2s},$$
$$\Pi_F^* = \frac{\theta_H^2\lambda}{4}\left(\frac{1}{s} + \frac{1}{c}\right) - \frac{cs(1-\lambda)u_0^2}{\theta_L^2(c+s)} - \frac{\lambda u_0\theta_H}{\theta_L}.$$

When $\frac{2csu_0(1-\lambda)}{\theta_L^2(c+s)} + \frac{\lambda\theta_H}{\theta_L} - 1 \leq 0$ *and* $\frac{(\theta_L - \theta_H\lambda)^2}{4c(1-\lambda)}\left(\frac{1}{s} + \frac{1}{c}\right) + \frac{\theta_H^2}{4s} \cdot (\lambda - \lambda^3) - u_0(1-\lambda) > 0$, *the firm launches two products with* $p^{H*} > p^{L*} > 0$. *The corresponding qualities, network effects, and profit are as follows:*

$$q^{L*} = \frac{\theta_L - \theta_H\lambda}{2c(1-\lambda)}, \ q^{H*} = \frac{\theta_H}{2c},$$
$$\alpha^{L*} = \frac{\theta_L - \theta_H\lambda}{2s(1-\lambda)}, \ \alpha^{H*} = \frac{\theta_H}{2s},$$
$$\Pi_{HL}^* = \frac{\theta_L^2 + \theta_H^2\lambda - 2\theta_L\theta_H\lambda}{4c(1-\lambda)} + \frac{\theta_H^2\lambda}{4s} + \frac{(\theta_L - \theta_H\lambda)^2}{4s(1-\lambda)} - u_0.$$

Otherwise, the firm provides only the high-end product, with price, quality, network effect, and profit as follows:

$$p^* = \frac{\theta_H^2}{2c} + \frac{\lambda^2\theta_H^2}{2s} - u_0,$$
$$q^* = \frac{\theta_H}{2c}, \ \alpha^* = \frac{\lambda\theta_H}{2s},$$
$$\Pi_H^* = \lambda\left(\frac{\theta_H^2}{4c} + \frac{\lambda^2\theta_H^2}{4s} - u_0\right).$$

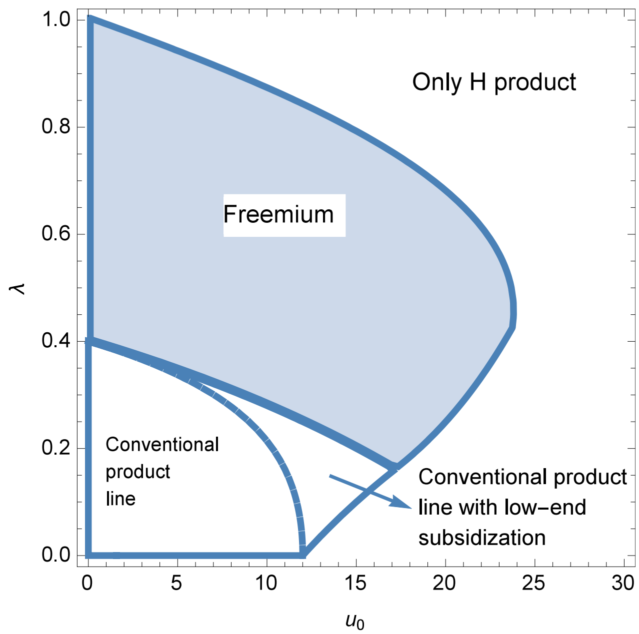
From Figure 3, we can see that under asymmetric network effects, the firm can indeed increase product-line profit by offering the low-end product for free. However, it should be noted that even when asymmetric network effects are present, freemium is not always an optimal strategy. When both λ and u_0 are relatively low, the firm segments the market via a conventional product line. In this case, p^L once again corresponds to the efficient price.

Given the above results, we compare the optimal quality decision under freemium and conventional product line design. We use the term "efficient quality," denoted by q^{i0*} with $i \in \{H, L\}$, to refer to the quality level that maximizes single product profit (therefore social welfare) under the complete information benchmark.

Corollary 4. *Across all equilibria, the quality of the high-end product is always set at the efficient level, that is,* $q^{H*} = q^{H0*}$;

- *when both segments are served with positive prices, the quality of the low-end product is always below the efficient level, that is,* $q^{L*} < q^{L0*}$;

Figure 3. (Color online) Strategies with Asymmetric Network Effects ($c = 0.5, s = 0.1, \theta_H = 5, \theta_L = 2$)



• *when the freemium strategy is adopted, the quality of the low-end product can be lower, equal to, or even greater than the efficient level, that is,* $q^{L*} = \frac{su_0}{\theta_L(c+s)} > q^{L0*}$ *can hold;*

• *the quality difference* $\Delta q = q^{H*} - q^{L*}$ *is smaller under the freemium strategy than that under conventional product line strategy with uniform network effects.*[11]

For freemium to be optimal, previous studies have recognized the importance of offering a balanced set of features in a firm's free product (Kumar 2014, Lee et al. 2017). Kumar (2014) stated that for freemium to work, the free offering has to be "compelling enough" to attract new users, but it cannot be "too rich" such that people stick to the free product. This insight is confirmed by our analysis. Whenever a freemium strategy is pursued, the firm has to strike a balance between getting consumers on board and minimizing cannibalization.

At the same time, this insight alone is not enough for the design of an optimal product line. Corollary 4 speaks to the importance of distinguishing between features that contribute to a product's standalone quality and features that contribute to its network effects. It states that when a firm pursues the freemium strategy, it should choose a standalone quality level for its low-end product that is higher than what it would choose in a conventional product line. In other words, a firm pursuing the freemium strategy should offer more features at a lower price (i.e., zero)! This reduces the quality differentiation within the product line. To compensate, the firm should design the products such that they offer different levels of network effects. In other words, it is not just the number of free features that determines freemium's viability but also which features are included in the free version. The prescription of higher standalone quality may seem counterintuitive at first, but a firm should realize that it is precisely a high standalone quality that allows the firm to choose lower network effects for the low-end product. Lower network effects are the key to minimizing cannibalization.

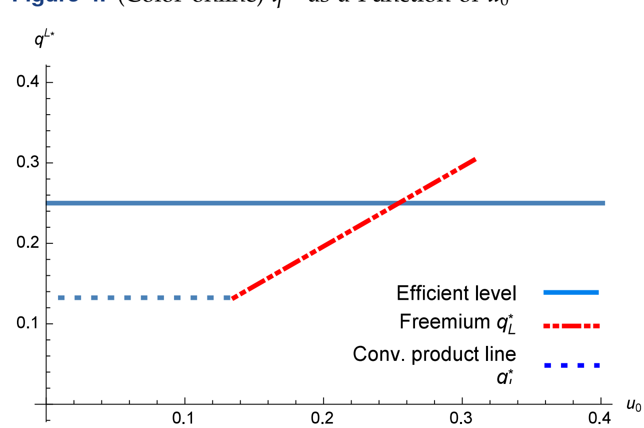
Figure 4 provides a simple illustration of this idea. It plots the equilibrium level of q^{L*} against the value of u_0. As u_0 increases, the equilibrium shifts from a conventional product line to the freemium equilibrium, and q^{L*} increases accordingly.

6. Concluding Remarks

This paper studies a monopolist's product and pricing strategy under network effects. We are particularly interested in the optimality of the freemium strategy. We seek to answer two questions. First, what are the necessary conditions for the optimality of freemium? Our results point to the asymmetry in network effects as the determining factor. Second, what are the principles that should guide the design of freemium? Our results add to the previous literature and speak to the distinction between a product's "baseline" quality and the network effects its users receive. Compared with a conventional product line, in a freemium equilibrium the firm should offer relatively higher baseline quality but low network effects on its low-end product.

Of course, as we have reviewed in Section 2, our analysis provides only one of many explanations of the optimality of freemium. Kamada and Ory (2015), for example, focused on the alternative explanation whereby freemium motivates word of mouth during the diffusion process. Other papers we reviewed in Section 2 considered the importance of consumer learning and free trial. In addition, competition may also play a key role in driving the price of the low-end product to the marginal cost. These explanations are clearly not mutually exclusive. Which explanation serves as the most likely explanation to the observed freemium depends on the context that is being considered. Our framework applies best to a scenario in which the product is beyond the

Figure 4. (Color online) q^{L*} as a Function of u_0



diffusion stage, and competition is not intense. When competition is relevant, the same mechanism may still be at work, but competition itself creates a downward pressure on the prices and provides the focal firm strong incentives to pull out the low-end product altogether. The exact effect is an interesting topic for future research.

Freemium has become an immensely popular business model among start-ups, especially in the Internet sector. Without doubt, providing a product for free is an effective way to expand a firm's user base. As many entrepreneurs have rightly believed, expanding a firm's user base is of ultimate importance for industries with network effects. However, our analysis points out that a firm should exercise caution when it is tempted to jump on the freemium bandwagon to "get users on board." For freemium to work, a firm has to understand the subtleties of network effects in its market. When network effects are not strong enough or are uniform across segments, offering an entry level product for free does no good to the product line profit. Freemium expands a firm's market share but severely limits its ability to create enough margin from its paid users. This scenario may sound familiar to many firms who are frustrated by the disappointing number of paying users under their freemium strategy. Instead, these firms should heed the wisdom of Mussa and Rosen (1978) and segment the market via a conventional product line. A conventional product line consists of a low-quality product that is sold at an efficient price instead of an entry-level product sold at zero price.

Our results in the extensions also provide guidelines to firms who should certainly adopt the freemium strategy. As previous studies have pointed out, a firm should provide an intermediate number of features in its free product. Moreover, it is not just the number of features that matters but also which features the firm should provide. Our analysis prescribes that a firm should in fact be quite generous with features that enhance a product's "baseline quality"—that is, the value of a product when it is used alone. However, the firm should deliberately limit the features that bring network benefits to users. In a well-designed freemium menu, "paying for upgrade" should in fact be "paying for network effects."

Our study focuses on a scenario wherein product line profit is the main driver of firm strategy. In doing so, we leave out many behavior factors that are nonetheless relevant to the freemium strategy. For example, offering a product for free can induce greater word of mouth and speed up its adoption (e.g., Kamada and Ory 2015). It would be interesting to combine the two perspectives and investigate the dynamics of the freemium strategy. This exercise may lead to a "taxonomy" of the freemium strategy and elucidate the possible motivations behind offering a free product. Second, it is interesting to extend the current model and examine the possibilities of advertising income. Advertising income should give the firm stronger incentives to provide the free product, and a firm should strike a balance between lower fees and higher ad revenue. Third, competition is a relevant factor in many high tech markets. Even though product-line competition brings considerable complexities to the model, it remains a meaningful direction for future research. Finally, it is of some interest to generalize the model into a case of user subsidization, in which negative price is possible. When the firm is able to subsidize the users, the price for the entry-level product is not constrained to be nonnegative. Although it is relatively easy to persuade adoption with subsidies, it is much harder to induce actual usage. It is of managerial interests to explore strategies that the firm could follow when network effects stem mostly from actual usage.

Appendix A. Proofs

Proof. Endnote 6.

Below we prove $\forall D, \frac{\partial[V^H(\theta,D)-V^L(\theta,D)]}{\partial\theta} > 0$ given the following two conditions:

$$\forall\theta, \frac{\partial[V^H(\theta,0)-V^L(\theta,0)]}{\partial\theta} > 0(*) \text{ and}$$
$$\forall\theta, D, \frac{\partial\left[\frac{\partial V^H(\theta,D)}{\partial D} - \frac{\partial V^L(\theta,D)}{\partial D}\right]}{\partial\theta} \geq 0(**).$$

We can write

$$V^H(\theta,D) - V^L(\theta,D) = \left[V^H(\theta,0) + \int_0^D \frac{\partial V^H(\theta,t)}{\partial t}dt\right] - \left[V^L(\theta,0) + \int_0^D \frac{\partial V^L(\theta,t)}{\partial t}dt\right]$$
$$= V^H(\theta,0) - V^L(\theta,0) + \int_0^D \frac{\partial[V^H(\theta,t) - V^L(\theta,t)]}{\partial t}dt$$

then given (*) and (**) we have

$$\frac{\partial[V^H(\theta,D) - V^L(\theta,D)]}{\partial\theta} = \frac{\partial\left[V^H(\theta,0) - V^L(\theta,0) + \int_0^D \frac{\partial[V^H(\theta,t)-V^L(\theta,t)]}{\partial t}dt\right]}{\partial\theta}$$
$$= \frac{\partial[V^H(\theta,0) - V^L(\theta,0)]}{\partial\theta} + \int_0^D \frac{\partial\left[\frac{\partial V^H(\theta,D)}{\partial D} - \frac{\partial V^L(\theta,D)}{\partial D}\right]}{\partial\theta}dt > 0. \quad \square$$

Proof. Lemma 1.

We used proof by contradiction for Lemma 1. Please refer to the main text for the complete proof. □

Proof. Proposition 1.

To prove Proposition 1, we show that when $\frac{\partial V^H(\theta,D)}{\partial D} = \frac{\partial V^L(\theta,D)}{\partial D} \geq 0$, the necessary condition for freemium to be optimal (i.e., $\frac{\partial\Pi_{HL}}{\partial p^L}|_{p^L=0} \leq 0$) is violated.

Let us now derive the general expression of Π as a piecewise function of p^H and p^L. Consider θ_L, θ_H and θ_{HL}, which are determined by

$$V^L(\theta_L, D) - p^L = u_0,$$
$$V^H(\theta_{HL}, D) - p^H = V^L(\theta_{HL}, D) - p^L,$$
$$V^H(\theta_H, D) - p^H = u_0.$$

Clearly, θ_L, θ_H and θ_{HL} are implicit functions of p^H and p^L. As a consequence, the profit function Π is a function of p^H and p^L:

$$\Pi = \begin{cases} p^H, \text{ when } p^H \text{ and } p^L \text{ are s.t. } \theta_L \le 0 \,\&\, \theta_{HL} \le 0. \\ \quad \text{In this case } D^L = 0, D^H = 1. \\ p^L, \text{ when } p^H \text{ and } p^L \text{ are s.t. } \theta_L \le 0 \,\&\, \theta_{HL} \ge 1. \\ \quad \text{In this case } D^L = 1, D^H = 0. \\ 0, \text{ when } p^H \text{ and } p^L \text{ are s.t. } \theta_L \ge 1 \,\&\, \theta_H \ge 1. \\ \quad \text{In this case } D^L = D^H = 0. \\ p^H D^H, \text{ when } p^H \text{ and } p^L \text{ are s.t. } \theta_L \ge 1 \,\&\, \theta_H \in (0,1), \\ \quad \textit{or } 0 < \theta_H < \theta_L \le 1. \\ \quad \text{In this case } D^L = 0, D^H = \int_{\theta_H}^1 f(\theta)d\theta. \\ p^L D^L, \text{ when } p^H \text{ and } p^L \text{ are s.t. } \theta_L \in (0,1) \,\&\, \theta_{HL} \ge 1. \\ \quad \text{In this case } D^L = \int_{\theta_L}^1 f(\theta)d\theta, D^H = 0. \\ p^L D^L + p^H D^H, \text{ when } p^H \text{ and } p^L \text{ are s.t. } \theta_L \le 0 \,\& \\ \quad \theta_{HL} \in (0,1), \textit{or } 0 < \theta_L < \theta_{HL} \le 1. \\ \quad \text{In this case } D^L = \int_{\theta_L \cdot I(\theta_L \ge 0)}^{\theta_{HL}} f(\theta)d\theta, \\ \quad D^H = \int_{\theta_{HL}}^1 f(\theta)d\theta. \end{cases}$$

$$= \begin{cases} p^H, \text{ when } p^H \text{ and } p^L \text{ are s.t. } \theta_L \le 0 \,\&\, \theta_{HL} \le 0. \\ p^L, \text{ when } p^H \text{ and } p^L \text{ are s.t. } \theta_L \le 0 \,\&\, \theta_{HL} \ge 1. \\ 0, \text{ when } p^H \text{ and } p^L \text{ are s.t. } \theta_L \ge 1 \,\&\, \theta_H \ge 1. \\ p^H \int_{\theta_H}^1 f(\theta)d\theta, \text{ when } p^H \text{ and } p^L \text{ are s.t. } \theta_L \ge 1 \\ \quad \& \, \theta_H \in (0,1), \\ \quad \textit{or } 0 < \theta_H < \theta_L \le 1. \\ p^L \int_{\theta_L}^1 f(\theta)d\theta, \text{ when } p^H \text{ and } p^L \text{ are s.t. } \theta_L \in (0,1) \\ \quad \& \, \theta_{HL} \ge 1. \\ p^L \int_{\theta_L \cdot I(\theta_L \ge 0)}^{\theta_{HL}} f(\theta)d\theta + p^H \int_{\theta_{HL}}^1 f(\theta)d\theta, \\ \quad \text{when } p^H \text{ and } p^L \text{ are s.t. } \theta_L \le 0 \,\&\, \theta_{HL} \in (0,1), \\ \quad \textit{or } 0 < \theta_L < \theta_{HL} \le 1. \end{cases}$$

First notice that, in equilibrium, θ_L^* cannot be smaller than 0, because the firm can always increase profit by increasing both p^H and p^L to make the type $\theta = 0$ indifferent between purchasing and not purchasing. Thus, we need to prove $\Pi_{HL} = p^L \int_{\theta_L}^{\theta_{HL}} f(\theta)d\theta + p^H \int_{\theta_{HL}}^1 f(\theta)d\theta$ violates the necessary condition $\frac{\partial \Pi_{HL}}{\partial p^L}|_{p^L=0} \le 0$, for any p^H as long as the demand schedule is $0 \le \theta_L < \theta_{HL} \le 1$.

Taking partial derivative of Π_{HL} w.r.t. p^L, we have

$$\frac{\partial \Pi_{HL}}{\partial p^L} = \int_{\theta_L}^{\theta_{HL}} f(\theta)d\theta + p^L[f(\theta_{HL}) - f(\theta_L)]\left(\frac{\partial \theta_{HL}}{\partial p^L} - \frac{\partial \theta_L}{\partial p^L}\right) - p^H f(\theta_{HL}) \frac{\partial \theta_{HL}}{\partial p^L}.$$

The sign of $\frac{\partial \Pi_{HL}}{\partial p^L}$ depends on the sign of $\frac{\partial \theta_L}{\partial p^L}$ and $\frac{\partial \theta_{HL}}{\partial p^L}$. Next we determine the signs of $\frac{\partial \theta_L}{\partial p^L}$ and $\frac{\partial \theta_{HL}}{\partial p^L}$. Recall from Equation (1), θ_L is implicitly given by $V^L(\theta_L, D) = p^L + u_0$. From Assumptions AII and AIII, we have $\frac{\partial V^i(\theta, D)}{\partial \theta} > 0$, $\frac{\partial V^i(\theta, D)}{\partial D} > 0$, and we also have D decreasing in p^L. Therefore, when p^L increases, we must have a higher θ_L to maintain the equality $V^L(\theta_L, D) = p^L + u_0$. In other words, we can get

$$\frac{\partial \theta_L}{\partial p^L} > 0. \tag{A.1}$$

Recall from Equation (2), θ_{HL} is implicitly given by $V^H(\theta_{HL}, D) - V^L(\theta_{HL}, D) = p^H - p^L$. When $\frac{\partial V^i(\theta, D)}{\partial D} = 0$ or $\frac{\partial V^H(\theta, D)}{\partial D} = \frac{\partial V^L(\theta, D)}{\partial D}$, we have $V^H(\theta_{HL}, D) - V^L(\theta_{HL}, D) = V^H(\theta_{HL}, 0) - V^L(\theta_{HL}, 0)$. Namely, the demand change does not affect the valuation differential. So we have $V^H(\theta_{HL}, 0) - V^L(\theta_{HL}, 0) = p^H - p^L$. From Assumption AIV, we have $\frac{\partial [V^H(\theta, 0) - V^L(\theta, 0)]}{\partial \theta} > 0$. Therefore when p^L decreases, we must have a higher θ_{HL} to maintain the equality $V^H(\theta_{HL}, 0) - V^L(\theta_{HL}, 0) = p^H - p^L$. In other words, we can get

$$\frac{\partial \theta_{HL}}{\partial p^L} < 0. \tag{A.2}$$

Because $\theta_{HL} > \theta_L$ and $f(\theta_{HL}) \ge 0$, with (A.2) we have

$$\frac{\partial \Pi_{HL}}{\partial p^L}|_{p^L=0} = \int_{\theta_L}^{\theta_{HL}} f(\theta)d\theta - p^H f(\theta_{HL}) \frac{\partial \theta_{HL}}{\partial p^L}|_{p^L=0} > 0. \quad \square$$

Proof. Corollary 1.

If the firm offers both products, according to Equations (1) and (2), we have

$$\theta_L = \frac{p^L + u_0 - \alpha}{q^L - \alpha},$$
$$\theta_{HL} = \frac{p^H - p^L}{q^H - q^L}.$$

With $\Pi_{HL} = p^L(\theta_{HL} - \theta_L) + p^H(1 - \theta_{HL})$, we have

$$\frac{\partial \Pi_{HL}}{\partial p^L} = \frac{2(p^H - p^L)}{q^H - q^L} - \frac{2p^L + u_0 - \alpha}{q^L - \alpha},$$
$$\frac{\partial \Pi_{HL}}{\partial p^H} = 1 - 2\frac{p^H - p^L}{q^H - q^L}.$$

Taking first-order conditions, we get

$$\theta_{HL}^* = \frac{1}{2}, \; \theta_L^* = \frac{q^L + u_0 - 2\alpha}{2(q^L - \alpha)},$$
$$p^{H*} = \frac{q^H - u_0}{2}, \; p^{L*} = \frac{q^L - u_0}{2}.$$

As long as $0 \le \theta_L^* < \theta_{HL}^*$, that is, $\begin{cases} q^L > \alpha > u_0 \\ q^L + u_0 \ge 2\alpha \end{cases}$, the above is the firm's optimal product and pricing strategy, that is, to offer both types of products, with $p^{H*} = \frac{q^H - u_0}{2}, p^{L*} = \frac{q^L - u_0}{2}$. Otherwise, the firm offers only the high-end product, with $p^{H*} = \frac{q^H - u_0}{2}$, $\theta_H^* = \frac{u_0 + q^H - 2\alpha}{2(q^H - \alpha)}$, and $\Pi_H^* = \frac{(q^H - u_0)^2}{4(q^H - \alpha)}$.[12] $\square$

Proof. Proposition 2.

The necessary condition for freemium to be optimal is $\frac{\partial \Pi_{HL}}{\partial p^L}|_{p^{L*}=0,p^{H*}} \leq 0$. We prove this can be satisfied when

$$\left[\frac{\partial V^H(\theta_{HL},D)}{\partial D} - \frac{\partial V^L(\theta_{HL},D)}{\partial D}\right]|_{p^{L*}=0,p^{H*}} \geq \left[\frac{1+\frac{\int_{\theta_L}^{\theta_{HL}} f(\theta)d\theta}{p^H f(\theta_{HL})}\left(\frac{\partial V^H(\theta_{HL},D)}{\partial \theta_{HL}} - \frac{\partial V^L(\theta_{HL},D)}{\partial \theta_{HL}}\right)}{f(\theta_L)\frac{\partial \theta_L}{\partial p^L}}\right]|_{p^{L*}=0,p^{H*}},$$

where the right-hand side is a positive value.

Recall from the proof of Proposition 1, we have

$$\frac{\partial \Pi_{HL}}{\partial p^L} = \int_{\theta_L}^{\theta_{HL}} f(\theta)d\theta + p^L[f(\theta_{HL}) - f(\theta_L)]\left(\frac{\partial \theta_{HL}}{\partial p^L} - \frac{\partial \theta_L}{\partial p^L}\right) - p^H f(\theta_{HL})\frac{\partial \theta_{HL}}{\partial p^L}.$$

According to the above equation, the necessary condition $\frac{\partial \Pi_{HL}}{\partial p^L}|_{p^{L*}=0,p^{H*}} \leq 0$ can be satisfied when

$$\frac{\partial \theta_{HL}}{\partial p^L}|_{p^{L*}=0,p^{H*}} \geq \frac{\int_{\theta_L}^{\theta_{HL}} f(\theta)d\theta}{p^H f(\theta_{HL})}|_{p^{L*}=0,p^{H*}}. \quad \text{(A.3)}$$

Next we determine the value of $\frac{\partial \theta_{HL}}{\partial p^L}|_{p^{L*}=0,p^{H*}}$.

Recall from Equation (2), we have $V^H(\theta_{HL},D) - p^H = V^L(\theta_{HL},D) - p^L$. By implicit function theorem, taking first derivative w.r.t. p^L on both sides, we can get

$$\frac{\partial V^H(\theta_{HL},D)}{\partial \theta_{HL}}\frac{\partial \theta_{HL}}{\partial p^L} + \frac{\partial V^H(\theta_{HL},D)}{\partial D}\frac{\partial D}{\partial p^L} = \frac{\partial V^L(\theta_{HL},D)}{\partial \theta_{HL}}\frac{\partial \theta_{HL}}{\partial p^L} + \frac{\partial V^L(\theta_{HL},D)}{\partial D}\frac{\partial D}{\partial p^L} - 1.$$

Rearranging, we have

$$\frac{\partial \theta_{HL}}{\partial p^L} = \frac{\left[\frac{\partial V^H(\theta_{HL},D)}{\partial D} - \frac{\partial V^L(\theta_{HL},D)}{\partial D}\right]\left(-\frac{\partial D}{\partial p^L}\right) - 1}{\frac{\partial V^H(\theta_{HL},D)}{\partial \theta_{HL}} - \frac{\partial V^L(\theta_{HL},D)}{\partial \theta_{HL}}}. \quad \text{(A.4)}$$

Substituting $\frac{\partial \theta_{HL}}{\partial p^L}$ expressed by Equation (A.4) into the inequality (A.3), we have

$$\left[\frac{\left[\frac{\partial V^H(\theta_{HL},D)}{\partial D} - \frac{\partial V^L(\theta_{HL},D)}{\partial D}\right]\left(-\frac{\partial D}{\partial p^L}\right) - 1}{\frac{\partial V^H(\theta_{HL},D)}{\partial \theta_{HL}} - \frac{\partial V^L(\theta_{HL},D)}{\partial \theta_{HL}}}\right]|_{p^{L*}=0,p^{H*}} \geq \frac{\int_{\theta_L}^{\theta_{HL}} f(\theta)d\theta}{p^H f(\theta_{HL})}|_{p^{L*}=0,p^{H*}}.$$

Rearranging, we have

$$\left[\frac{\partial V^H(\theta_{HL},D)}{\partial D} - \frac{\partial V^L(\theta_{HL},D)}{\partial D}\right]|_{p^{L*}=0,p^{H*}} \geq \left[\frac{1+\frac{\int_{\theta_L}^{\theta_{HL}} f(\theta)d\theta}{p^H f(\theta_{HL})}\left(\frac{\partial V^H(\theta_{HL},D)}{\partial \theta_{HL}} - \frac{\partial V^L(\theta_{HL},D)}{\partial \theta_{HL}}\right)}{f(\theta_L)\frac{\partial \theta_L}{\partial p^L}}\right]|_{p^{L*}=0,p^{H*}}.$$

Notice that the above rearrangement can be got because $\frac{\partial[V^H(\theta_{HL},D) - V^L(\theta_{HL},D)]}{\partial \theta_{HL}} > 0$ and $\frac{\partial D}{\partial p^L}|_{p^L=0} < 0$. The former is given by Assumption AIV. Here we show the latter. Following the same logic in the proof of Proposition 1, when $\frac{\partial V^L(\theta,D)}{\partial D} \neq \frac{\partial V^H(\theta,D)}{\partial D}$, we still have $\frac{\partial \theta_L}{\partial p^L} > 0$. For Π_{HL}, we have $D = \int_{\theta_L}^1 f(\theta)d\theta$, and $\frac{\partial D}{\partial p^L}|_{p^L=0} = -f(\theta_L)\frac{\partial \theta_L}{\partial p^L} < 0$. With $\frac{\partial \theta_L}{\partial p^L} > 0$, $f(\theta) > 0$, and $\frac{\partial[V^H(\theta_{HL},D) - V^L(\theta_{HL},D)]}{\partial \theta_{HL}} > 0$ (AIV), it follows that $\left[\frac{1+\frac{\int_{\theta_L}^{\theta_{HL}} f(\theta)d\theta}{p^H f(\theta_{HL})}\left(\frac{\partial V^H(\theta_{HL},D)}{\partial \theta_{HL}} - \frac{\partial V^L(\theta_{HL},D)}{\partial \theta_{HL}}\right)}{f(\theta_L)\frac{\partial \theta_L}{\partial p^L}}\right]|_{p^{L*}=0,p^{H*}}$ is a positive value. □

Proof. Corollary 2 and Corollary 3.

According to Lemma 1, we have

$$p^L = \theta_L(q^L - \alpha^L) + \alpha^L - u_0,$$
$$p^H = (q^H - q^L)\theta_{HL} + \alpha^H + \theta_L(q^L - \alpha^H) - u_0.$$

The total profit is

$$\begin{aligned}\Pi_{HL}(\theta_{HL},\theta_L) &= p^H(1-\theta_{HL}) + p^L(\theta_{HL} - \theta_L)\\ &= [(q^H - q^L)\theta_{HL} + \alpha^H + \theta_L(q^L - \alpha^H) - u_0]\\ &\quad \cdot (1-\theta_{HL})\\ &\quad + [\theta_L(q^L - \alpha^L) + \alpha^L - u_0](\theta_{HL} - \theta_L)\\ \text{s.t.} \quad & 0 \leq \theta_L < \theta_{HL} < 1,\\ & \theta_L(q^L - \alpha^L) + \alpha^L - u_0 \geq 0.\end{aligned}$$

We can see that $\Pi_{HL}(\theta_{HL},\theta_L)$ is concave in both θ_{HL} and θ_L. Therefore, we can get the global optimal $\theta^*_{HL}, \theta^*_L$ that maximize Π_{HL}, by employing the first derivatives of Π_{HL} w.r.t. θ_{HL}, θ_L, respectively. Then p^{L*} and p^{H*} can be obtained.

$$\frac{\partial \Pi_{HL}}{\partial \theta_{HL}} = \theta_L(\alpha^H - \alpha^L) + \alpha^L - \alpha^H + (q^H - q^L)(1 - 2\theta_{HL}),$$
$$\begin{aligned}\frac{\partial \Pi_{HL}}{\partial \theta_L} &= (q^L - \alpha^H)(1-\theta_{HL}) + (q^L - \alpha^L)(\theta_{HL} - 2\theta_L) - \alpha^L + u_0\\ &= \theta_{HL}(\alpha^H - \alpha^L) - 2\theta_L(q^L - \alpha^L) + q^L - \alpha^H - \alpha^L + u_0.\end{aligned}$$

Alternatively, we can write the marginal consumer types as $\theta_L = \frac{p^L + u_0 - \alpha^L}{q^L - \alpha^L}$, $\theta_{HL} = \frac{p^H - p^L - (\alpha^H - \alpha^L)\frac{q^L - p^L - u_0}{q^L - \alpha^L}}{q^H - q^L}$, and the total demand is $D = 1 - \theta_L$. With $0 < \theta_L < 1$ and $0 < \theta_H < 1$, we must have $p^L + u_0 < q^L$ and $q^L > \alpha^L$. We can express the profit as

$$\begin{aligned}\Pi_{HL}(p^L,p^H) &= p^L(\theta_{HL} - \theta_L) + p^H(1-\theta_{HL})\\ &= p^L\left[\frac{(p^H - p^L)(q^L - \alpha^L) - (\alpha^H - \alpha^L)(q^L - p^L - u_0)}{(q^H - q^L)(q^L - \alpha^L)} - \frac{p^L + u_0 - \alpha^L}{q^L - \alpha^L}\right]\\ &\quad + p^H\left[1 - \frac{(p^H - p^L)(q^L - \alpha^L) - (\alpha^H - \alpha^L)(q^L - p^L - u_0)}{(q^H - q^L)(q^L - \alpha^L)}\right].\end{aligned}$$

When $0 \le \frac{p^{L*}+u_0-\alpha^L}{q^L-\alpha^L} < \frac{p^{H*}-p^{L*}-(\alpha^H-\alpha^L)\frac{q^L-p^{L*}-u_0}{q^L-\alpha^L}}{q^H-q^L} < 1$, $0 < p^{L*} < p^{H*}$, and $\Pi^*_{HL} \ge \frac{(q^H-u_0)^2}{4(q^H-\alpha^H)}$, the firm would offer two products with positive prices and

$$\theta^*_L = \frac{(\alpha^H-\alpha^L)(\alpha^H-\alpha^L-q^H+q^L) + 2(q^H-q^L)(\alpha^H+\alpha^L-u_0-q^L)}{(\alpha^H-\alpha^L)^2-4(q^H-q^L)(q^L-\alpha^L)},$$

$$\theta^*_{HL} = \frac{(\alpha^H-\alpha^L)^2+\alpha^L(u_0+2q^H-3q^L)+\alpha^H(q^L-u_0) - 2q^L(q^H-q^L)}{(\alpha^H-\alpha^L)^2-4(q^H-q^L)(q^L-\alpha^L)},$$

$$\Pi^*_{HL} = \frac{(q^H-q^L)[\alpha^L(q^H-u_0)-\alpha^H(q^L-u_0)+(2u_0-q^H)q^L-u_0^2]}{(\alpha^H-\alpha^L)+4(\alpha^L-q^L)(q^H-q^L)}.$$

When $\frac{\partial \Pi_{HL}}{\partial p^L}\big|_{p^{L*}=0, p^{H*}>0} \le 0$, $0 < \frac{u_0-\alpha^L}{q^L-\alpha^L} < \frac{(q^H-q^L)q^L-\alpha^L(q^H-2q^L+u_0)+\alpha^H(u_0-q^L)}{2(q^H-q^L)(q^L-\alpha^L)} < 1$, and $\Pi^*_{HL} \ge \frac{(q^H-u_0)^2}{4(q^H-\alpha^H)}$, the firm offers two products with:

$p^{L*} = 0$, $p^{H*} = \frac{q^L(\alpha^H-q^L)-(\alpha^H-\alpha^L)u_0}{2(q^L-\alpha^L)} + \frac{q^H}{2}$,

$\theta^*_L = \frac{u_0-\alpha^L}{q^L-\alpha^L}$, $\theta^*_{HL} = \frac{(q^H-q^L)q^L-\alpha^L(q^H-2q^L+u_0)+\alpha^H(u_0-q^L)}{2(q^H-q^L)(q^L-\alpha^L)}$,

$\Pi^*_{HL} = \frac{[\alpha^L(q^H-u_0)-q^L(q^H-q^L)+\alpha^H(u_0-q^L)]^2}{4(q^H-q^L)(q^L-\alpha^L)^2}$.

Otherwise the firm offers only the high-end product with $p^{H*} = \frac{q^H-u_0}{2}$, $\theta^*_H = \frac{q^H+u_0-2\alpha^H}{2(q^H-\alpha^H)}$, $\Pi^*_H = \frac{(q^H-u_0)^2}{4(q^H-\alpha^H)}$. □

Proof. Concavity of profit function in q^i.

When q^i is an endogenous decision, we prove that, if $V^i(\theta, D)$ is concave in q^i and $C(q^i)$ is convex in q^i for $i \in \{H, L\}$, the profit function is concave in q^i.

First, we look at the case in which only the high-end product is provided. By Lemma 1, we have θ_H defined by $V^H(\theta_H, D) - p^H = u_0$, so $p^H = V^H(\theta_H, D) - u_0$.

$$\Pi_H = [p^H - C(q^H)]\int_{\theta_H}^1 f(\theta)d\theta = [V^H(\theta_H, D) - u_0 - C(q^i)]\int_{\theta_H}^1 f(\theta)d\theta,$$

$$\frac{\partial \Pi_H}{\partial q^H} = \left[\frac{\partial V^H(\theta_H, D)}{\partial q^H} - \frac{\partial C(q^i)}{\partial q^H}\right]\int_{\theta_H}^1 f(\theta)d\theta,$$

$$\frac{\partial^2 \Pi_H}{\partial (q^H)^2} = \left[\frac{\partial^2 V^H(\theta_H, D)}{\partial (q^H)^2} - \frac{\partial^2 C(q^i)}{\partial (q^H)^2}\right]\int_{\theta_H}^1 f(\theta)d\theta.$$

Therefore, when $V^i(\theta, D)$ is concave in q^i, and $C(q^i)$ is convex in q^i for $i \in \{H, L\}$, we have $\frac{\partial^2 \Pi_H}{\partial (q^H)^2} < 0$; thus, Π_H is concave in q^H.

Following the same logic, we can get Π_{HL} is concave in q^H and q^L. □

Proof. Proposition 3.

In conventional product line design

$$\Pi_{HL} = (1-\theta_{HL})[p^H - c(q^H)^2] + (\theta_{HL}-\theta_L)[p^L - c(q^L)^2]$$
$$\text{s.t. } p^L = \theta_L q^L + \alpha^L(1-\theta_L) - u_0,$$
$$p^H = \theta_{HL}(q^H - q^L) + (\alpha^H - \alpha^L)(1-\theta_L) + p^L,$$
$$0 < \theta_L < \theta_{HL} < 1.$$

Because Π_{HL} is concave in q^i, with first-order conditions with respect to q^i, we have:

$$\begin{cases} q^{H*} = \frac{\theta_{HL}}{2c} \\ q^{L*} = \frac{\theta_{HL}+\theta_L-1}{2c} \end{cases}.$$

Substituting q^{H*} and q^{L*} into Π_{HL}, we can derive the optimal decision of $\theta^*_{HL}, \theta^*_L$:

$$\begin{cases} \theta^*_{HL} = \frac{1}{15}\left(11 + 8\alpha^L c - \sqrt{1 - 64\alpha^L c + 64(\alpha^L)^2c^2 + 60cu_0}\right) \\ \theta^*_L = \frac{1}{15}\left(7 + 16\alpha^L c - 2\sqrt{1 - 64\alpha^L c + 64(\alpha^L)^2c^2 + 60cu_0}\right) \end{cases}$$

or

$$\begin{cases} \theta^*_{HL} = \frac{1}{15}\left(11 + 8\alpha^L c + \sqrt{1 - 64\alpha^L c + 64(\alpha^L)^2c^2 + 60cu_0}\right) \\ \theta^*_L = \frac{1}{15}\left(7 + 16\alpha^L c + 2\sqrt{1 - 64\alpha^L c + 64(\alpha^L)^2c^2 + 60cu_0}\right) \end{cases}$$

subject to $0 < \theta^*_L < \theta^*_{HL} < 1$. Substituting into q^{H*} and q^{L*}, we can get $\frac{\partial q^{L*}}{\partial \alpha^L} > 0$.

When freemium is optimal, we have

$$\Pi_F = (1-\theta_{HL})\left[p^H - c(q^H)^2\right] + (\theta_{HL}-\theta_L)\left[-c(q^L)^2\right]$$
$$\text{s.t. } p^L = 0 = \theta_L q^L + \alpha^L(1-\theta_L) - u_0,$$
$$p^H = \theta_{HL}(q^H - q^L) + (\alpha^H - \alpha^L)(1-\theta_L),$$
$$0 < \theta_L < \theta_{HL} < 1$$

because Π_F is concave in q^i. With first-order conditions with respect to q^H, we have $q^{H*} = \frac{\theta_{HL}}{2c}$. Taking derivative w.r.t. q^L, we have $\frac{\partial \Pi_F}{\partial q^L} < 0$ always holds.

$$\begin{cases} q^{H*} = \frac{\theta_{HL}}{2c} \\ q^{L*} = \frac{u_0-\alpha^L(1-\theta_L)}{\theta_L} \end{cases}. \tag{A.5}$$

With (A.5), we have

$$\frac{\partial q^{L*}}{\partial \alpha^L} = \frac{\theta_L - 1}{\theta_L} < 0. \quad \square$$

Proof. Proposition 4.

With $\alpha^H = \alpha^L = \alpha$, we now have

$$V^L(\theta, \alpha, D) = \theta(q^L + \alpha D),$$
$$V^H(\theta, \alpha, D) = \theta(q^H + \alpha D),$$

where D is the total demand of all offered products. As is typical in games with network effects, multiple equilibria may exist in the second stage. We seek the Nash equilibrium that is Pareto dominant. More specifically, when network effects are intermediate, there exist multiple equilibria wherein all, some, or none of the consumers adopt the products. When consumers do not adopt, the products do not generate sufficient network effects, and thus nonadoption becomes self-fulfilling. This coordination failure is classic in models with network effects. Clearly, the equilibrium wherein all users adopt generates (weakly) higher surplus for all parties. Thus, we select that equilibrium whenever it exists. For the proof of Proposition 4, two cases are analyzed below.

Case 1. Sell to Both Segments with $(q^H, \alpha, p^H), (q^L, \alpha, p^L)$.

When network effects are present, the binding constraints in the firm's optimization problem continue to be the low-end consumers' individual rationality (IR) constraint and the high-end consumers' incentive compatibility (IC) constraint. The optimal prices satisfy:

$$p^L = \theta_L(q^L + \alpha) - u_0,$$
$$p^H = \theta_H(q^H - q^L) + \theta_L(q^L + \alpha) - u_0.$$

We bound q^L above zero in the following analysis. The optimal qualities can therefore be determined by a profit maximizing problem wherein:

$$\Pi_{HL} = \lambda\Big[p^H - c(q^H)^2 - s\alpha^2\Big] + (1-\lambda)\Big[p^L - c(q^L)^2 - s\alpha^2\Big] \text{ s.t. } p^L \geq 0, q^L \geq 0.$$

Using the Lagrangian method, the optimal menu is

$$q^{H*} = \frac{\theta_H}{2c},$$

$$q^{L*} = \begin{cases} \frac{\theta_L - \theta_H\lambda}{2c(1-\lambda)}, \lambda\theta_H\theta_L \leq \theta_L^2 \;\&\; \frac{s[\theta_H\theta_L\lambda + 2cu_0(1-\lambda)]}{\theta_L^2(c - c\lambda + s)} - 1 \leq 0 \\ 0, \lambda\theta_H\theta_L > 2su_0 > \theta_L^2 \\ 0, \lambda\theta_H\theta_L > \theta_L^2 \geq 2su_0 \\ \frac{2su_0\theta_L - \theta_H\lambda}{2\theta_L(c - c\lambda + s)}, 2su_0 > \lambda\theta_H\theta_L \;\&\; 2su_0 > \theta_L^2 \;\&\; \frac{s[\theta_H\theta_L\lambda + 2cu_0(1-\lambda)]}{c - c\lambda + \theta_L^2 s} - 1 > 0 \end{cases},$$

$$\alpha^* = \begin{cases} \frac{\theta_L}{2s}, \lambda\theta_H\theta_L \leq \theta_L^2 \;\&\; \frac{s[\theta_H\theta_L\lambda + 2cu_0(1-\lambda)]}{\theta_L^2(c - c\lambda + s)} - 1 \leq 0 \\ u_0, \lambda\theta_H\theta_L > 2su_0 > \theta_L^2 \\ \frac{\theta_L}{2s}, \lambda\theta_H\theta_L > \theta_L^2 \geq 2su_0 \\ \frac{\theta_H\theta_L\lambda + 2cu_0(1-\lambda)}{2\theta_L(c - c\lambda + s)}, 2su_0 > \lambda\theta_H\theta_L \;\&\; 2su_0 > \theta_L^2 \;\&\; \frac{s[\theta_H\theta_L\lambda + 2cu_0(1-\lambda)]}{c - c\lambda + \theta_L^2 s} - 1 > 0 \end{cases},$$

$$p^{L*} = \begin{cases} \frac{\theta_L(\theta_L - \theta_H\lambda)}{2c(1-\lambda)} + \frac{\theta_L^2}{2s} - u_0, \lambda\theta_H\theta_L \leq \theta_L^2 \;\&\; \frac{s[\theta_H\theta_L\lambda + 2cu_0(1-\lambda)]}{\theta_L^2(c - c\lambda + s)} - 1 \leq 0 \\ 0, \lambda\theta_H\theta_L > 2su_0 > \theta_L^2 \\ \frac{\theta_L^2}{2s} - u_0, \lambda\theta_H\theta_L > \theta_L^2 \geq 2su_0 \\ 0, 2su_0 > \lambda\theta_H\theta_L \;\&\; 2su_0 > \theta_L^2 \;\&\; \frac{s[\theta_H\theta_L\lambda + 2cu_0(1-\lambda)]}{c - c\lambda + \theta_L^2 s} - 1 > 0 \end{cases},$$

$$p^{H*} = \begin{cases} \frac{\theta_H^2 + \theta_L^2 - \theta_H\theta_L(1+\lambda)}{2c(1-\lambda)} + \frac{1}{2s} - u_0, \lambda\theta_H\theta_L \leq \theta_L^2 \;\&\; \frac{s[\theta_H\theta_L\lambda + 2cu_0(1-\lambda)]}{\theta_L^2(c - c\lambda + s)} - 1 \leq 0 \\ \frac{\theta_H^2}{2c}, \lambda\theta_H\theta_L > 2su_0 > \theta_L^2 \\ \frac{\theta_H^2}{2c} + \frac{\theta_L^2}{2s} - u_0, \lambda\theta_H\theta_L > \theta_L^2 \geq 2su_0 \\ \theta_H\left(\frac{\theta_H}{2c} - \frac{2su_0 - \theta_H\theta_L\lambda}{2\theta_L(c - c\lambda + s)}\right), 2su_0 > \lambda\theta_H\theta_L \;\&\; 2su_0 > \theta_L^2 \;\&\; \frac{s[\theta_H\theta_L\lambda + 2cu_0(1-\lambda)]}{c - c\lambda + \theta_L^2 s} - 1 > 0 \end{cases},$$

$$\Pi_{HL}^* = \begin{cases} \frac{\theta_L^2 + \theta_H^2\lambda - 2\theta_H\theta_L\lambda}{4c(1-\lambda)} + \frac{\theta_L^2}{4s} - u_0, \lambda\theta_H\theta_L \leq \theta_L^2 \;\&\; \frac{s[\theta_H\theta_L\lambda + 2cu_0(1-\lambda)]}{\theta_L^2(c - c\lambda + s)} - 1 \leq 0 \\ \frac{\theta_H^2\lambda}{4c} - \frac{su_0^2}{\theta_L^2}, \lambda\theta_H\theta_L > 2su_0 > \theta_L^2 \\ \frac{\theta_H^2\lambda}{4c} + \frac{\theta_L^2}{4s} - u_0, \lambda\theta_H\theta_L > \theta_L^2 \geq 2su_0 \\ \frac{\theta_H^2\theta_L^2\lambda(c+s) - 4\theta_H\theta_L scu_0\lambda - 4sc^2u_0^2(1-\lambda)}{4\theta_L^2 c(c + s - c\lambda)}, 2su_0 > \lambda\theta_H\theta_L \;\&\; 2su_0 > \theta_L^2 \;\&\; \frac{s[\theta_H\theta_L\lambda + 2cu_0(1-\lambda)]}{c - c\lambda + \theta_L^2 s} - 1 > 0. \end{cases}.$$

Case 2. Sell to θ_H-Segment Only With(q, α, p).

Binding condition $p = \theta_H(q + \alpha\lambda) - u_0$. Firm's profit is given by $\Pi_H = \lambda(p - cq^2 - s\alpha^2)$. Solving the optimization problem, we obtain:

$$p^* = \frac{\theta_H^2}{2c} + \frac{\lambda^2\theta_H^2}{2s} - u_0, \; q^* = \frac{\theta_H}{2c}, \alpha^* = \frac{\lambda\theta_H}{2s},$$
$$\Pi_H^* = \lambda\left(\frac{\theta_H^2}{4c} + \frac{\lambda^2\theta_H^2}{4s} - u_0\right).$$

Freemium is optimal if and only if (IFF) the conditions $\Pi_{HL}^* \geq \Pi_H^*, p^{L*} = 0$ are satisfied.

Below we prove the above conditions cannot hold simultaneously. According to the results obtained by using the Lagrangian method, we discuss by parameter ranges where p^{L*} may possibly be zero.

1. When $\lambda\theta_H\theta_L > 2su_0 > \theta_L^2$.

We have

$$\Pi_H^* - \Pi_{HL}^* = \lambda\left(\frac{\theta_H^2\lambda^2}{4s} - u_0\right) + \frac{su_0^2}{\theta_L^2} > \lambda\left(u_0\frac{su_0}{\theta_L^2} - u_0\right) + \frac{u_0}{2} > u_0\left(\frac{1}{2} - \frac{\lambda}{2}\right) > 0.$$

Therefore, $\Pi_{HL}^* < \Pi_H^*$ always holds when $\lambda\theta_H\theta_L > 2su_0 > \theta_L^2$; thus, the firm prefers to offer only the high-end product, and freemium cannot emerge.

2. When $2su_0 > \lambda\theta_H\theta_L, 2su_0 > \theta_L^2$ and $\frac{s[\theta_H\theta_L\lambda + 2cu_0(1-\lambda)]}{c - c\lambda + \theta_L^2 s} - 1 > 0$.

Following the same logic as in (1), we prove $\Pi_H^* > \Pi_{HL}^*$ always holds as long as the high-type consumer finds the low-end product worth trying at price zero, that is, $\theta_L(q^L + \lambda\alpha) - u_0 \geq 0$.

$$\Pi_{HL}^* = \frac{\theta_H^2\theta_L^2\lambda(c+s) - 4\theta_H\theta_L scu_0\lambda - 4sc^2u_0^2(1-\lambda)}{4\theta_L^2 c(c + s - c\lambda)} = \frac{\lambda\theta_H^2}{4c} + \frac{\theta_H^2\lambda^2}{4(c + s - c\lambda)} - \frac{su_0[\theta_H\theta_L\lambda + cu_0(1-\lambda)]}{\theta_L^2(c + s - c\lambda)},$$
$$\Pi_H^* = \frac{\lambda\theta_H^2}{4c} + \lambda\left(\frac{\lambda^2\theta_H^2}{4s} - u_0\right).$$

Let $F_{HL} = \frac{\theta_H^2\lambda^2}{4(c+s-c\lambda)} - \frac{su_0[\theta_H\theta_L\lambda+cu_0(1-\lambda)]}{\theta_L^2(c+s-c\lambda)}$, $F_H = \lambda\,(\frac{\lambda^2\theta_H^2}{4s} - u_0)$. The IFF condition for $\Pi_H^* > \Pi_{HL}^*$ is $F_H > F_{HL}$. We proceed by proving that, under the condition $p^{L*} < 0, F_H > F_{HL(max)}$ holds; thus, $F_H > F_{HL}$ and $\Pi_H^* > \Pi_{HL}^*$.

Below we prove F_{HL} is decreasing in c; thus, $\sup_{c>0} F_{HL} = \lim_{c\to 0} F_{HL} = F_{HL}|_{c=0} = \frac{\theta_H^2\lambda^2}{4s} - \frac{u_0\theta_H\lambda}{\theta_L}$. Therefore, a sufficient condition for $F_H > F_{HL}$ is $F_H > \sup_{c>0} F_{HL}$, or equivalently, $F_H > F_{HL}|_{c=0}$. (Notice F_H is not a function of c.)

$$\frac{\partial F_{HL}}{\partial c} = -\frac{\theta_H^2\theta_L^2\lambda(1-\lambda)}{4\theta_L^2(c+s-c\lambda)} - \frac{4su_0(su_0-\theta_H\theta_L\lambda)(1-\lambda)}{4\theta_L^2(c+s-c\lambda)}$$
$$= -\frac{(1-\lambda)[4su_0(su_0-\theta_H\theta_L\lambda)+\theta_H^2\theta_L^2\lambda]}{4\theta_L^2(c-c\lambda+s)}.$$

We have $4su_0(su_0-\theta_H\theta_L\lambda)+\theta_H^2\theta_L^2\lambda > 4\frac{\lambda\theta_H\theta_L}{2}(\frac{\lambda\theta_H\theta_L}{2} - \theta_H\theta_L\lambda) + \theta_H^2\theta_L^2\lambda = -\theta_H^2\theta_L^2\lambda^2 + \theta_H^2\theta_L^2\lambda > 0$, hence $\frac{\partial F_{HL}}{\partial c} < 0$, and F_{HL} is decreasing in c.

Next we prove $F_H > F_{HL}|_{c=0}$. When c approaches 0, the condition $\frac{s[\theta_H\theta_L\lambda+2cu_0(1-\lambda)]}{\theta_L^2(c-c\lambda+s)} - 1 > 0$ implies $\lambda > \frac{\theta_L}{\theta_H}$. We first prove $F_{HL}|_{c=0} < 0$ for all $\lambda > \frac{\theta_L}{\theta_H}$.

At $\lambda > \frac{\theta_L}{\theta_H}$, we have

$$F_{HL}|_{c=0,\lambda=\frac{\theta_L}{\theta_H}} = \frac{\theta_L^2}{4s} - u_0 < \frac{2su_0}{4s} - u_0 = -\frac{u_0}{2} \le 0$$

and

$$\frac{\partial F_{HL}|_{c=0}}{\partial\lambda} = \frac{\theta_H^2\lambda}{2s} - \frac{u_0\theta_H}{\theta_L} < \frac{2su_0/\theta_L}{2s}\theta_H - \frac{u_0\theta_H}{\theta_L} = 0.$$

Therefore, when $\lambda > \frac{\theta_L}{\theta_H}$, we always have $F_{HL}|_{c=0} < 0$.

We also have

$$F_H = \lambda\left(\frac{\lambda^2\theta_H^2}{4s} - u_0\right)$$
$$> \lambda\left(\frac{\lambda^2\theta_H^2}{4s} - u_0\frac{\theta_H}{\theta_L}\lambda\right)$$
$$= \lambda F_{HL}|_{c=0}$$
$$> F_{HL}|_{c=0}\ (\text{since}\ F_{HL}|_{c=0} < 0).$$

So we have proved $F_H > F_{HL}|_{c=0}$; hence, $F_H > F_{HL}$ is also proved. □

Proof. Proposition 5.

The proof follows similar logic as in Proposition 4. With asymmetric network effects, consumer valuations of the products are:

$$V^L(\theta,\alpha^L,D) = \theta(q^L + \alpha^L D),$$
$$V^H(\theta,\alpha^H,D) = \theta(q^H + \alpha^H D).$$

Case 1. Sell to Both Segments with (q^H,α^H,p^H), (q^L,α^L,p^L).

The optimal prices satisfy:

$$p^L = \theta_L(q^L + \alpha^L) - u_0,$$
$$p^H = \theta_H(q^H - q^L + \alpha^H - \alpha^L) + \theta_L(q^L + \alpha^L) - u_0.$$

The optimal qualities can therefore be determined by a profit maximizing problem wherein:

$$\Pi_{HL} = \lambda\left[p^H - c(q^H)^2 - s(\alpha^H)^2\right] + (1-\lambda)\left[p^L - c(q^L)^2 - s(\alpha^L)^2\right]\ \text{s.t.}\ p^L \ge 0, q^L \ge 0.$$

The optimal quality levels remain the same as in the no-network-effects scenario, namely

$$q^{H*} = \frac{\theta_H}{2c},$$
$$q^{L*} = \begin{cases} \frac{\theta_L-\theta_H\lambda}{2c(1-\lambda)}, & \frac{2csu_0(1-\lambda)}{\theta_L^2(c+s)} + \frac{\lambda\theta_H}{\theta_L} - 1 \le 0 \\ \frac{su_0}{\theta_L(c+s)}, & \frac{2csu_0(1-\lambda)}{\theta_L^2(c+s)} + \frac{\lambda\theta_H}{\theta_L} - 1 > 0 \end{cases},$$
$$\alpha^{H*} = \frac{\theta_H}{2s},$$
$$\alpha^{L*} = \begin{cases} \frac{\theta_L-\theta_H\lambda}{2s(1-\lambda)}, & \frac{2csu_0(1-\lambda)}{\theta_L^2(c+s)} + \frac{\lambda\theta_H}{\theta_L} - 1 \le 0 \\ \frac{cu_0}{\theta_L(c+s)}, & \frac{2csu_0(1-\lambda)}{\theta_L^2(c+s)} + \frac{\lambda\theta_H}{\theta_L} - 1 > 0 \end{cases},$$
$$p^{L*} = \begin{cases} \frac{\theta_L(\theta_L-\theta_H\lambda)}{2(1-\lambda)}\left(\frac{1}{c}+\frac{1}{s}\right) - u_0, & \frac{2csu_0(1-\lambda)}{\theta_L^2(c+s)} + \frac{\lambda\theta_H}{\theta_L} - 1 \le 0 \\ 0, & \frac{2csu_0(1-\lambda)}{\theta_L^2(c+s)} + \frac{\lambda\theta_H}{\theta_L} - 1 > 0 \end{cases},$$
$$p^{H*} = \begin{cases} \frac{\theta_H^2+\theta_L^2-(1+\lambda)\theta_L\theta_H}{2(1-\lambda)}\left(\frac{1}{c}+\frac{1}{s}\right) - u_0, & \frac{2csu_0(1-\lambda)}{\theta_L^2(c+s)} + \frac{\lambda\theta_H}{\theta_L} - 1 \le 0 \\ \frac{\theta_H^2}{2}\left(\frac{1}{c}+\frac{1}{s}\right) - \frac{u_0\theta_H}{\theta_L}, & \frac{2csu_0(1-\lambda)}{\theta_L^2(c+s)} + \frac{\lambda\theta_H}{\theta_L} - 1 > 0 \end{cases}.$$

The optimal profit is:

$$\Pi_{HL}^* = \begin{cases} \frac{\theta_H^2+\theta_L^2\lambda-2\lambda\theta_L\theta_H}{4(1-\lambda)}\left(\frac{1}{c}+\frac{1}{s}\right) - u_0, & \frac{2csu_0(1-\lambda)}{\theta_L^2(c+s)} + \frac{\lambda\theta_H}{\theta_L} - 1 \le 0 \\ \frac{\theta_H^2\lambda(c+s)}{4cs} - \frac{\theta_H\lambda u_0}{\theta_L} - \frac{csu_0^2(1-\lambda)}{\theta_L^2(c+s)}, & \frac{2csu_0(1-\lambda)}{\theta_L^2(c+s)} + \frac{\lambda\theta_H}{\theta_L} - 1 > 0 \end{cases}.$$

Case 2. Sell to θ_H-Segment Only.

Here the situation is the same as in Case 2 in the symmetric network effects scenario, where $p = \theta_H(q + \alpha\lambda) - u_0$ and $\Pi_H = \lambda\,(p - cq^2 - s\alpha^2)$:

$$q^* = \frac{\theta_H}{2c}, \alpha^* = \frac{\lambda\theta_H}{2s},$$
$$\Pi_H^* = \lambda\left(\frac{\theta_H^2}{4c} + \frac{\lambda^2\theta_H^2}{4s} - u_0\right).$$

For freemium to be optimal, the conditions $\Pi_{HL}^* \ge \Pi_H^*, p^{L*} = 0$ have to be satisfied.

Under asymmetric network effects, freemium equilibrium exists, when (s, c, u_0, λ) satisfy the conditions below:

$$\frac{\theta_H^2}{4s}(\lambda - \lambda^3) + \lambda u_0\left(1 - \frac{\theta_H}{\theta_L}\right) - \frac{cs(1-\lambda)u_0^2}{\theta_L^2(c+s)} \ge 0,$$
$$\frac{\theta_H^2\lambda(c+s)}{4sc} - \frac{cs(1-\lambda)u_0^2}{\theta_L^2(c+s)} - \frac{\theta_H\lambda u_0}{\theta_L} > 0,$$
$$\frac{2csu_0(1-\lambda)}{\theta_L^2(c+s)} + \frac{\lambda\theta_H}{\theta_L} - 1 > 0.$$

It can be seen that the above inequalities define a nonempty set. Corollary 4 follows straightforwardly according to the above results. □

Appendix B

In the analysis in Section 5.2, we alluded to the possibility that the assumption "the high-type consumers have positive valuation for the low-end product at price zero" may not hold. This means that the highest-type consumers will not find the low-end product worth trying even if it is offered at zero price, which is very unlikely for successful freemium products. In this section we provide a detailed analysis for this exceptional case and show that even if the assumption is violated, our results still hold true unless the cost function has some special form. With some special cost function and the assumption violated (i.e., $V^L(\theta_H, \alpha, \lambda) < 0$), raising the low-end product's price above zero will violate the high type's *IR* instead of *IC* constraint, then it is possible that freemium would emerge under uniform network effects.

As in the baseline model, the firm can either serve the high-type consumers with menu (p, q, α_1), getting profit Π; or serve both high and low segments, with menu $(p^H, q^H, \alpha_2), (p^L, q^L, \alpha_2)$, getting profit Π_{HL}. Freemium is a special case of the second strategy, whereby $p^L = 0$ and profit Π_F. We focus on necessary conditions for freemium to be optimal.

Let $C_q(q)$, $C_\alpha(\alpha)$ be the marginal cost for offering one product of quality q and network benefit α.

1. When $\lambda > \frac{\theta_L}{\theta_H}$.

By checking only p and p^H, we can see that the firm prefers offering only the high-end product to adopting freemium strategy. More specifically, consider p^{H*}:

$$\begin{aligned} p^{H*} &= \theta_H(q^{H*} - q^{L*}) + \theta_L(q_L^* + \alpha_2^*) - u_0 \\ &< \theta_H(q^{H*} - q^{L*}) + \theta_H q_L^* + \theta_L \alpha_2^* - u_0 \\ &= \theta_H q^{H*} + \theta_L \alpha_2^* - u_0. \end{aligned}$$

Now let us consider p, where we let the firm sets $\alpha_1 = \alpha_2^*$:

$$\begin{aligned} p &= \theta_H(q^* + \lambda\alpha_2^*) - u_0 \ (IRconstraint) \\ &= \theta_H q^{H*} + \theta_H \lambda \alpha_2^* - u_0 \ (since q^* = q^{H*}) \\ &> \theta_H q^{H*} + \theta_L \alpha_2^* - u_0 \ (since \lambda > \frac{\theta_L}{\theta_H}) \\ &> p^{H*}. \end{aligned}$$

Therefore, compared with serving only the high-type consumers, it is never optimal to pursue the freemium strategy, because the firm will get less profit on the high end while subsidizing the low-end segment. In fact, when selling only to the high-type customers, the firm can even increase profit by adopting an optimal α_1^*, making $\Pi^* > \Pi_{HL}^*$ always hold under $\lambda > \frac{\theta_L}{\theta_H}$.

2. When $\lambda \le \frac{\theta_L}{\theta_H}$.

First, consider serving only the high end. We have:

$$\begin{aligned} p &= \theta_H(q + \lambda\alpha_1) - u_0, \\ \Pi &= \lambda[p - C_q(q) - C_\alpha(\alpha_1)] \\ &= \lambda[\theta_H(q + \lambda\alpha_1) - u_0 - C_q(q) - C_\alpha(\alpha_1)]. \end{aligned}$$

To maximize Π, we have:

$$\begin{aligned} C_q{}'(q^*) &= \theta_H, \\ C_\alpha{}'(\alpha_1^*) &= \lambda\theta_H \end{aligned}$$

for $\Pi^* > 0$, we have $u_0 < \theta_H(q^* + \lambda\alpha_1^*) - C_q(q^*) - C_\alpha(\alpha_1^*)$.

Then, when the firm serves both segments, we have:

$$\begin{aligned} p^H &= \theta_H(q^H - q^L) + \theta_L(q^L + \alpha_2) - u_0, \\ p^L &= \theta_L(q^L + \alpha_2) - u_0, \\ \Pi_{HL} &= \lambda[p^H - C_q(q^H)] + (1-\lambda)[p^L - C_q(q^L)] - C_\alpha(\alpha_2). \end{aligned}$$

To maximize Π_{HL}, we have:

$$\begin{aligned} C_q{}'(q^{H*}) &= \theta_H, \\ C_q{}'(q^{L*}) &= \frac{\theta_L - \lambda\theta_H}{1-\lambda}, \\ C_\alpha{}'(\alpha_2^*) &= \theta_L. \end{aligned}$$

Notice the necessary condition for freemium is $p^{L*} \le 0$, where $p^{L*} = \theta_L(q^{L*} + \alpha_2^*) - u_0$. Under this condition, we check whether the firm would like to induce more cost (by offering higher-than-optimal $q^L + \alpha_2$ to make $p^L = 0$) to have the low-type consumers on board. Supposing the firm adopts freemium, we need to compare Π^* and Π_F, where the firm set q^L and α_2 such that $p_L = \theta_L(q^L + \alpha_2) - u_0 = 0$ and $q^{H*} > q^L \ge q^{L*}$. Notice that $q^* = q^{H*}$, and q^{H*} is always equal to the efficient quality.

Assuming a convex cost function, with $u_0 < \theta_H(q^* + \lambda\alpha_1^*) - C_q(q^*) - C_\alpha(\alpha_1^*)$, we have:

$$\begin{aligned} \Delta\Pi &= \Pi_F^* - \Pi^* \\ &= \lambda(u_0 - \alpha_1^*\lambda\theta_H - \theta_H q^L) - (1-\lambda)C_q(q^L) \\ &\quad + \lambda C_\alpha(\alpha_1^*) - C_\alpha(\alpha_2) \\ &< \lambda(\theta_H q^{H*} - C_q(q^*) - \theta_H q^L) - (1-\lambda)C_q(q^L) - C_\alpha(\alpha_2) \\ &< \lambda[C_q{}'(q^{H*})(q^{H*} - q^L) - C_q(q^{H*})] - (1-\lambda)C_q(q^L). \end{aligned}$$

We can see that $\Delta\Pi < 0$ unless $C_q(\cdot)$ is very steep and skewed towards zero, specifically, $C_q{}'(q^{H*}) > \left(\frac{1-\lambda}{\lambda}\right)\frac{C_q(q^L)}{q^{H*}-q^L} + \frac{C_q(q^{H*})}{q^{H*}-q^L}$.

Above we analyzed the case in which network effect is endogenous. As can be easily seen from the analysis, exactly the same conclusion can be reached for the case in which network effects are exogenously given, no matter how large α is.

Endnotes

[1] See Zetlin (2013), for example, an interview of Rhapsody's chief executive officer, who insisted on its subscription-only model while competitors adopted freemium. In the gaming industry, leading firms such as Blizzard Inc. offer freemium on some of their games but not others.

[2] Throughout the paper we use the term "efficient quality" to refer to the quality level that maximizes single product profit (therefore social welfare) under the complete information benchmark.

[3] A number of papers have touched upon this issue (Cheng and Tang 2010, Niculescu and Wu 2014), but none of them have completely endogenized prices and qualities in the product line.

[4] For example, keeping a product for free saves the need to set up a payment system, potentially lowering a firm's operating cost. Similarly, zero price has been shown to be a particularly powerful marketing tool and might be preferred to a small but positive price for behavioral reasons.

[5] In the main text, we consider cases in which outside options for all consumers are the same. In the online appendices, we provide

analysis for the cases in which consumers have heterogeneous outside options. In short, heterogeneous outside options do not qualitatively affect our results in the main analysis.

[6] These two requirements together are sufficient conditions of Assumption AIV; please see Appendix A for proof.

[7] Note that it is not an equilibrium wherein the firm sells only the low-end product. Comparatively, offering only the low-end product is dominated and trivial, because the firm can always attract the same user bases with higher price by offering the high-end product. Therefore, selling only the low-end product is trivially dominated.

[8] We thank an evaluator for this insightful comment.

[9] In Appendix B, we explain specifically for the discrete case in which this assumption does not hold, which means that the highest-type consumers will not find the low-end product worth trying even if it is offered at zero price. This is implausible if not impossible according to those successful freemium products offered in the market.

[10] The exact expressions of optimal prices are provided in Appendix A.

[11] Here we compare the quality difference, assuming that the relevant parameter values are the same.

[12] Note our implicit assumption is that the firm has incentive to enter the market, thus $\Pi_H^* > 0$, that is, $q^H > \alpha$.

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